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On Infinitesimal Projective and Conformal Transformations in Riemannian Spaces

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ABSTRACT

The main objective of the paper is to study infinitesimal projective and conformal transformation in Riemannian spaces. We have obtained the conditions for which the Lie derivatives of conformal curvature tensor, W_2 curvature tensors vanishes under infinitesimal projective and conformal transformations. If the metric of a manifold is Ricci solution and if it admits an infinitesimal transformation then we have proved that the manifold is Einstein.

Keywords: Projective and Conformal Transformation; Ricci Soliton.

1. Introduction

Let M be an n-dimensional connected Riemannian manifold with Riemannian metric g. Let R^h_{ijk} (resp. R_{ij} , R) be the components of Riemannian curvature tensor (resp. The Ricci tensor, The scalar curvature) of the manifold M. Let us consider in M^n an infinitesimal transformation

(1.1) $$\bar{x}^i = x^i + \xi^i \delta_t$$

Where ξ^i is a vector field in M^n. We denote by £ the Lie derivative with respect to ξ^i. The transformation (1.1) is a projective transformation if and only if

(1.2) $$£\Gamma^i_{jk} = \delta^i_j p_k + \delta^i_k p_j$$

Where p_i is a gradient. If p_i (1.2) is zero, then the infinitesimal transformation (1.1) is an affine one. Again, if (1.1) is a conformal transformation if and only if

(1.3) $$£g_{ij} = \varphi g_{ij}$$

Where φ is a non-constant function of x's and it is a motion when φ in (1.3) is zero.

In 1962, M. Prvanovitch [6] studied infinitesimal projective and conformal transformations in recurrent and Ricci-recurrent Riemannian spaces. In 1966, W. Roter [9] and again, in 1983, R.K. Garai and H. Sen [3] studied this type of transformations.

In this paper we have studied infinitesimal projective and conformal transformations on projective curvature tensor, W_2-curvature tensor, conharmonic curvature tensor and Weyl conformal curvature tensor.

2. Infinitesimal Projective Transformations

If M^n admits a non-affine infinitesimal projective transformation for which

$$£\Gamma^i_{jk} = \delta^i_j p_k + \delta^i_k p_j$$

Then we have the following [10]:

(2.1) $\pounds R^h_{ijk} = \delta^h_j p_{i,k} - \delta^h_k p_{i,j}$,

(2.2) $\pounds R_{ij} = (1-n) p_{i,j}$,

(2.3) $\pounds P^h_{ijk} = 0$,

Where Γ is the Christoffel symbol of g_{ij} and P is the projective curvature tensor i.e. [2]

(2.4) $P^h_{ijk} = R^h_{ijk} - \frac{1}{n-1}(\delta^h_k R_{ij} - \delta^h_j R_{ik})$

In 1970 G. P. Pokhariyal and R. S. Mishra [5] introduced the notion of a new curvature tensor, denoted by W_2 and studied its relativistic significance. The W_2-curvature tensor of type (1,3) is defined by

(2.5) $(W_2)^h_{ijk} = R^h_{ijk} + \frac{1}{n-1}(\delta^h_k R_{ij} - \delta^h_j R_{ik})$

Taking the Lie derivative with respect to the field ξ^i to the above, we have

(2.6) $\pounds (W_2)^h_{ijk} = \pounds R^h_{ijk} + \frac{1}{n-1}(\delta^h_k \pounds R_{ij} - \delta^h_j \pounds R_{ik})$

Using (2.1) and (2.2) in (2.6), we get

(2.7) $\pounds (W_2)^h_{ijk} = \delta^h_i p_{j,k} - \delta^h_k p_{i,j}$,

If p_i is parallel in M^n, i.e., $p_{i,j}$=0, then from above we have $\pounds (W_2)^h_{ijk} = 0$.

Again if $\pounds (W_2)^h_{ijk} = 0$ then from (2.7) we have

$\delta^h_i p_{j,k} - \delta^h_k p_{i,j} = 0$

Contracting h and k in above, we obtain

$(1-n) p_{i,j} = 0$

i.e., p_i is parallel.

Hence we can state the following:

Theorem 2.1. *If a M^n admits an infinitesimal projective transformation (1.1) for which* $\pounds \Gamma^i_{jk} = \delta^i_j p_k + \delta^i_k p_j$, then $\pounds (W_2)^h_{ijk} = 0$ if and only if p_i is parallel in M^n.

The conformal curvature tensor of type (1.3) is defined by

(2.8) $C^h_{ijk} = R^h_{ijk} - \frac{1}{n-2}(\delta^h_k R_{ij} - \delta^h_j R_{ik} + R^h_k g_{ij} - R^h_j g_{ik}) - \frac{R}{(n-1)(n-2)}(\delta^h_k g_{ij} - \delta^h_j g_{ik})$

As a special subgroup of the conformal transformation group, Y. Ishii [4] introduced the notion of the conharmonic transformation under which a harmonic function transforms into a harmonic function. The conharmonic curvature tensor $\bar{C}$ of type (1,3) on a Riemannian manifold (M^n,g), (n>3), (this condition is assumed as for n= 3 the Weyl conformal tensor vanishes) is given by

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(2.9) $\bar{C}^h_{ijk} = R^h_{ijk} - \frac{1}{n-2}(\delta^h_k R_{ij} - \delta^h_j R_{ik} + R^h_k g_{ij} - R^h_j g_{ik})$.

Taking the Lie derivative with respect to the field ξ^i to the above, we have

(2.10) $£\bar{C}^h_{ijk} = £R^h_{ijk} - \frac{1}{n-2}[\delta^h_k £R_{ij} - \delta^h_j £R_{ik} + £(R^h_k g_{ij}) - £(R^h_j g_{ik})]$.

If $£\bar{C}^h_{ijk} = 0$, then from (2.10) we have

(2.11) $£R^h_{ijk} = \frac{1}{n-2}[\delta^h_k £R_{ij} - \delta^h_j £R_{ik} + £(R^h_k g_{ij}) - £(R^h_j g_{ik})]$

Using (2.1) and (2.2) in (2.11), we get

$\delta^h_i p_{j,k} - \delta^h_k p_{i,j} = £(R^h_k g_{ij}) - £(R^h_j g_{ik})$

Contracting *h* and k in above and using (2.2), we obtain

$£(Rg_{ij}) = 0$.

Again, if $£(Rg_{ij}) = 0$.

Then from (2.10), we have $£\bar{C}^h_{ijk} = 0$. Hence we have the following:

Theorem 2.2. *If a M^n admits an infinitesimal projective transformation (1.1) for which* $£\Gamma^i_{jk} = \delta^i_j p_k + \delta^i_k p_j$, then $£\bar{C}^h_{ijk} = 0$ if and only if $£(Rg_{ij}) = 0$.

If $£\bar{C}^h_{ijk} = 0$, (2.11) holds. Now taking the Lie derivative of (2.8) and using (2.11). We have

(2.12) $£C^h_{ijk} = \frac{1}{(n-1)(n-2)}[[\delta^h_k £(Rg_{ij}) - [\delta^h_j £(Rg_{ik})]$

Since $£\bar{C}^h_{ijk} = 0$ if and only if $£(Rg_{ij}) = 0$. From (2.12) we have

$£C^h_{ijk} = 0$.

Theorem 2.3. *If a M^n admits an infinitesimal projective transformation (1.1) for which* $£\Gamma^i_{jk} = \delta^i_j p_k + \delta^i_k p_j$, then the following conditions are equivalent:

(i) $£\bar{C}^h_{ijk} = 0$

(ii) $£(Rg_{ij}) = 0$

(iii) $£C^h_{ijk} = 0$.

3. Infinitesimal Conformal Transformations

If M^n admits an infinitesimal conformal transformation for which $£g_{ij} = 2\varphi g_{ij}$, then we have [10]:

(3.1) $£\Gamma^i_{jk} = \delta^i_j \varphi_k + \delta^i_k \varphi_j - \varphi^i g_{ij}$

(3.2) $£g^{ij} = -2\varphi g^{ij}$

and

(3.3) $\pounds R^h_{ijk} = \delta^h_j \varphi_{i,k} - \delta^h_k \varphi_{i,j} + \varphi^h_{,j} g_{ik} - \varphi^h_{,k} g_{ij}$

Contracting *h* and *k* in (3.3), we have

(3.4) $\pounds g_{ij} = (2-n)\delta^h_k \varphi_{i,j} - \varphi^t_{,t} g_{ij}$

From (3.2) and (3.4), it follows that

(3.5) $\pounds R = \pounds g^{ij} R_{ij} = -2[\varphi R + (n-1)\varphi^t_{,t}$

and

(3.6) $\pounds R^i_j = \pounds g^{ik} R_{kj} = -2\varphi R^i_j + (2-n)\varphi^i_{,j} - \delta^i_j \varphi^t_{,t}$

In an infinitesimal conformal transformation, $\pounds C^h_{ijk} = 0$, where *C* is the Weyl conformal curvature tensor given in (2.8).

Also in an infinitesimal conformal transformation, $\pounds P^h_{ijk} = 0$, where P is the projective curvature tensor given in (2.4).

Now taking the Lie derivative with respect to the field ξ^i to the equation (2.5), we get

(3.7) $\pounds (W_2)^h_{ijk} = \pounds R^h_{ijk} + \frac{1}{n-1}(\delta^h_j \pounds R_{ik} - \delta^h_i \pounds R_{jk})$

If $\pounds (W_2)^h_{ijk} = 0$, then from above we get

$$\pounds R^h_{ijk} + \frac{1}{n-1}(\delta^h_j \pounds R_{ik} - \delta^h_i \pounds R_{jk}) = 0$$

By using (3.3) and (3.3) to the above, we get

$$\delta^h_j \varphi_{i,k} - \delta^h_k \varphi_{i,j} + \varphi^h_{,j} g_{ik} - \varphi^h_{,k} g_{ij} = \frac{2-n}{n-1}(\delta^h_i \varphi_{j,k} - \delta^h_j \varphi_{i,k} + \frac{1}{n-1}(\delta^h_j \varphi^t_{,t} g_{ik} - \delta^h_i \varphi^t_{,t} g_{jk})$$

Contracting *h* and *k* to the above, we obtain

$(2-n)\varphi_{i,j} = \varphi^t_{,t} g_{ij}$, which implies

$\varphi_{i,j} = \frac{2}{n-1}\varphi^t_{,t} g_{ij}$.

Again if $\varphi_{i,j} = \frac{2}{n-1}\varphi^t_{,t} g_{ij}$, then from (3.7), and by using (3.3) and (3.3), we get

$\pounds (W_2)^h_{ijk} = 0$.

Hence we can state the following:

Theorem 3.1. *If a* M^n *admits an infinitesimal projective transformation (1.1) for which*

$\pounds \Gamma^i_{jk} = \delta^i_j \varphi_k + \delta^i_k \varphi_j - \varphi^i g_{ij}$, then $\pounds (W_2)^h_{ijk} = 0$.

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If and only if $\varphi_{i,j} = \frac{2}{n-1}\varphi^t_{,t}g_{ij}$.

Applying the Lie derivative with respect to the vector field ξ^I to (2.9), we obtain

(3.8) $\pounds\bar{C}^h_{ijk} = \pounds R^h_{ijk} - \frac{1}{n-2}[\delta^h_k \pounds R_{ij} - \delta^h_j \pounds R_{ik} + \pounds(R^h_k g_{ij}) - \pounds(R^h_j g_{ik})]$.

If $\pounds\bar{C}^h_{ijk} = 0$, then from above

(3.9) $\pounds R^h_{ijk} = \frac{1}{n-2}[\delta^h_k \pounds R_{ij} - \delta^h_j \pounds R_{ik} + \pounds(R^h_k g_{ij}) - \pounds(R^h_j g_{ik})]$.

By using (3.2)-(3.6) in (3.9), we get

(3.10) $\delta^h_j \varphi_{i,k} - \delta^h_k \varphi_{i,j} + \varphi^h_{,j} g_{ik} - \varphi^h_{,k} g_{ij}$

$= \frac{1}{n-2}[\delta^h_k (2-n)\varphi_{i,j} - \delta^h_k \varphi^t_{,t} g_{ij} - \delta^h_j (2-n)\varphi_{i,k} + \delta^h_j \varphi^t_{,t} g_{ik}$

$+ R^h_k \pounds g_{ij} + g_{ij}\pounds R^h_k - R^h_j \pounds g_{ik} + g_{ik}\pounds R^h_j$

$+ (2-n)\varphi^h_{,k} g_{ij} + \delta^h_k \varphi^t_{,t} g_{ij} - (2-n)\varphi^h_{,j} g_{ik} - \delta^h_j \varphi^t_{,t} g_{ik}]$.

Again contracting h and k, we get

(3.11) $(2-n)\varphi_{i,j} - \varphi^t_{,t} g_{ij} = (2-n)\varphi_{i,j} - \frac{3n-4}{n-2}\varphi^t_{,t} g_{ij}$, which implies

$g_{ij} = 0$.

Again, if we take $\varphi^t_{,t} g_{ij} = 0$, then by using (3.2)-(3.6), we get from (3.8), $\pounds\bar{C}^h_{ijk} = 0$.

This leads to the following:

Theorem 3.2. *If a M^n admits an infinitesimal projective transformation (1.1) for which*

$\pounds\Gamma^i_{jk} = \delta^i_j \varphi_k + \delta^i_k \varphi_j - \varphi^i g_{ij}$, then $\pounds\bar{C}^h_{ijk} = 0$ if and only if $\varphi^t_{,t} g_{ij} = 0$.

4. Ricci Soliton under Infinitesimal Conformal Transformation

In, 1982, Hamilton [1] introduced the notion of Ricci flow to find a canonical metric on a smooth manifold. Then Ricci flow has become a powerful tool for the study of Riemannian manifolds, especially for those manifolds with positive curvature. Perelman ([7], [8]) used Ricci flow and its surgery to prove Poincare conjecture. The Ricci flow is an evolution equation for metrics on a Riemannian manifold defined as follows:

$\frac{\delta}{\delta t} g_{ij}(t) = -2R_{ij}$

A Ricci soliton emerges as the limit of the solutions of the Ricci flow. A solution to the Ricci flow is called Ricci soliton if it moves only by a one parameter group of diffeomorphism and scaling. A Ricci soliton (g, V, λ) on a Riemannian manifold (M, g) is a generalization of an Einstein metric such that [1]

(4.1) $\pounds_V g + 2S + 2\lambda g = 0$.

If a M^n admits am infinitesimal conformal transformation then $\pounds g = 2\varphi g$. Putting in (1.1) we get

(4.2) $S = -(\varphi g + \lambda g)$.

That is the manifold is Einstein. Thus we have the following:

Theorem 4.1. Let the metric of a Riemannian manifold is Ricci soliton. If the manifolds admit an infinitesimal conformal transformation, then the manifold is Einstein.

Corollary 4.1. Let the metric of a Riemannian manifold is Ricci soliton. If the manifolds admit an infinitesimal conformal transformation, then the function φ is constant.

Where S is the Ricci tensor, $\pounds_V$ is the Lie derivative operator along the vector field V.

Declarations

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Competing Interests Statement

The author has declared no competing interests.

Consent for Publication

The author declares that he consented to the publication of this study.

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