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On the Sextic Diophantine Equation with Five Unknowns $2(x+y)(x^3-y^3)=61(z^2-w^2)p^4$

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A B S T R A C T

We obtain infinitely many non-zero integer quintuples (x, y, z, w, p) satisfying the non-homogenous sextic equation with five unknowns. Various interesting properties among the values of x, y, z, w and p are presented. Some relations between the solutions and special numbers are exhibited.

Keywords: Integral solutions, Sextic equation with five unknowns and Special numbers.

MSC Classification: 11D41

1. INTRODUCTION

The theory of Diophantine equations offers a rich variety of fascinating problems [1-4]. Particularly in [5,6], sextic equations with three unknowns are studied for their integral solutions; [7-12] analyze sextic equations with 4 unknowns for their non-zero integer solutions and [13-15] deals with sextic equation with 5 unknowns.

This communication analyses sextic equation with five unknowns given by

$$2(x+y)(x^3-y^3)=61(z^2-w^2)p^4$$

Infinitely many non-zero integer quintuples satisfying the above equation are obtained. Various interesting properties among the values of x, y, z, w and p are presented.

2. NOTATIONS USED

- $t_{m,n}$ - Polygonal number of rank n with size m.
- P_n^m - Pyramidal number of rank n with size m.
- Pr_n - Pronic number of rank n
- J_n - Jacobsthal number of rank n.
- $CP_{m,n}$ - Centered Pyramidal Number.
- $F_{4,s}^{r}$ - Fourth dimensional Figurate number of rank n.

3. METHOD OF ANALYSIS

The diophantine equation to be solved for its non-zero distinct integral solutions is given by

$$2(x+y)(x^3-y^3)=61(z^2-w^2)p^4 \quad (1)$$

Introducing the transformations

$$x=u+v, y=u-v,$$
$$z=u+2v, w=u-2v \;,\; u\neq v\neq 0 \quad (2)$$

In (1), it leads to

$$v^2+3u^2=61p^4 \quad (3)$$

(3) can be solved through different methods and we obtain different sets of integer solutions to (1).

Set 1:

Assume $p=a^2+3b^2$ (4)

Where a and b are non-zero distinct integers.

Write 61 as $61=(7+i2\sqrt{3})(7-i2\sqrt{3})$ (5)

Substituting (4) & (5) in (3) and applying the method of factorization, define

$$v+i\sqrt{3}u=(7+i2\sqrt{3})(a+i\sqrt{3}b)^4$$

Equating the real and imaginary parts, we have

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$$u = 2a^4 + 18b^4 - 36a^2b^2 + 28a^3b - 84ab^3$$
$$v = 7a^4 + 63b^4 - 126a^2b^2 - 24a^3b + 72ab^3 \quad (6)$$

From (2), the integer solutions of (1), are

$$x(a,b) = 9a^4 + 81b^4 - 162a^2b^2 + 4a^3b - 12ab^3$$
$$y(a,b) = -5a^4 - 45b^4 + 90a^2b^2 + 52a^3b - 156ab^3$$
$$z(a,b) = 16a^4 + 144b^4 - 288a^2b^2 - 20a^3b + 60ab^3$$
$$w(a,b) = -12a^4 - 108b^4 + 216a^2b^2 + 76a^3b - 228ab^3$$
$$p(a,b) = a^2 + 3b^2$$

Properties:

- $y(a,1) - 13x(a,1) + 122\left[t_{4,a^2} - 18t_{4,a} + (J_3)^2\right] = 0.$
- $x(a,1) + y(a,1) - 4t_{4,a^2} - 112P_a^5 + 128t_{4,a} \equiv 36 \pmod{168}.$
- $6\{y(a,a) - x(a,a) - p(a,a) + 4t_{4,a}\}$ is a nasty number.
- $z(1,b) + w(1,b) - 36t_{4,b^2} + 168CP_{6,b} + 72\,\mathrm{Pr}_b \equiv 4 \pmod{128}.$
- $\{y(3,1) - x(3,1)\}$ is the sum of two squares.
- Each of the following expressions represents a bi-quadratic integer:

$$8\{w(a,a) - z(a,a)\}$$
$$\{y(a,a) - x(a,a)\}$$

Set 2:

One may write (3) as

$$v^2 + 3u^2 = 61p^4 * 1 \quad (7)$$

Write 1 as

$$1 = \frac{(1+i\sqrt{3})(1-i\sqrt{3})}{4} \quad (8)$$

Using (4) , (5) and (8) in (7) and applying the method of factorization and equating the real parts, we get

$$(v + i\sqrt{3}u) = \frac{1}{2}\left[(1+i\sqrt{3})(a+i\sqrt{3}b)^4(7+i2\sqrt{3})\right] \quad (9)$$

Equating real and imaginary parts of (9), we have

$$u = \frac{1}{2}(9a^4 + 81b^4 - 162a^2b^2 + 4a^3b - 12ab^3)$$
$$v = \frac{1}{2}(a^4 + 9b^4 - 18a^2b^2 - 108a^3b + 324ab^3)$$

As our aim is to find integer solutions, choosing a=2A , b=2B in the above equations we obtain

$$u = 72A^4 + 648B^4 - 1296A^2B^2 + 32A^3B - 96AB^3$$
$$v = 8A^4 + 72B^4 - 144A^2B^2 - 864A^3B + 2592AB^3 \quad (10)$$
$$p = 4A^2 + 12B^2 \quad (11)$$

In view of (2), the integer solutions of (1) are given by

$$x(A,B) = 80A^4 + 720B^4 - 1440A^2B^2 - 832A^3B + 2496AB^3$$
$$y(A,B) = 64A^4 + 576B^4 - 1152A^2B^2 + 896A^3B - 2688AB^3$$
$$z(A,B) = 88A^4 + 792B^4 - 1584A^2B^2 - 1696A^3B + 5088AB^3$$
$$w(A,B) = 56A^4 + 504B^4 - 1008A^2B^2 + 1760A^3B - 5280AB^3$$
$$p(A,B) = 4A^2 + 12B^2$$

Properties:

- $x(A,1) + y(A,1) - 64CP_{6,A} + 5184t_{3,A} - a\ perfect\ square \equiv 1296 \pmod{2592}.$

- $y(A,1)-64\begin{bmatrix}12F^r_{4,4}+10CP_{6,A}\\-23t_{4,A}-44A\end{bmatrix}$ is a perfect square.
- $z(A,1)-w(A,1)-32t_{4,A^2}+3456CP_{6,A}+a\ perfect\ square\equiv 288(\text{mod } 10368).$
- $x(A,1)+y(A,1)-t_{4,12A^2-36}-64CP_{6,A}+1728t_{4,A}\equiv 0(\text{mod } 5184).$
- $\{x(1,1)-y(1,1)\}$ is the sum of two squares.

Note:
Equation (8), can also be written as

$$1=\frac{(1+i4\sqrt{3})(1-i4\sqrt{3})}{49} \qquad (12)$$

Proceeding as above, the different sets of integer solutions of (1) are illustrated below:

$$x(A,B)=4459A^4+40131B^4-80262A^2B^2-146804A^3B+440412AB^3$$
$$y(A,B)=16121A^4+145089B^4-290178A^2B^2+100156A^3B-300468AB^3$$
$$z(A,B)=-1372A^4-12348B^4+24696A^2B^2-270284A^3B+810852AB^3$$
$$w(A,B)=21952A^4+197568B^4-395136A^2B^2+223636A^3B-670908AB^3$$
$$p(A,B)=49A^2+147B^2$$

Remark:
Instead of (2), one may also introduce another set of transformations as

$$x=u+v, y=u-v, z=2uv+1, w=2uv-1, \quad (u\neq v\neq 0) \qquad (13)$$

For this choice, the corresponding sets of distinct integer solutions to (1) are as represented below:

Set 3:
By substituting the equation (4) and (6) in (13) we obtain the integral solutions to (1) are given by

$$x(a,b)=9a^4+81b^4-162a^2b^2+4a^3b-12ab^3$$
$$y(a,b)=-5a^4-45b^4+90a^2b^2+52a^3b-156ab^3$$
$$z(a,b)=28[a^8+81b^8-84a^6b^2-756a^2b^6+630a^4b^4]+296[a^7b-27ab^7-21a^5b^3+63a^3b^5]+1$$
$$w(a,b)=28[a^8+81b^8-84a^6b^2-756a^2b^6+630a^4b^4]+296[a^7b-27ab^7-21a^5b^3+63a^3b^5]-1$$
$$p(a,b)=a^2+3b^2$$

Set 4:
And also by substituting the equation (10) and (11) in (13) we obtain the integral solutions to (1) are given by

$$x(A,B)=80[A^4+9B^4-18A^2B^2]-832[A^3B-3AB^3]$$
$$y(A,B)=64[A^4+9B^4-18A^2B^2]+896[A^3B-3AB^3]$$
$$z(A,B)=1152[A^8+81B^8-84A^6B^2+756A^2B^6+630A^4B^4]-123904[A^7B-27AB^7-21A^5B^3+63A^3B^5]+1$$
$$w(A,B)=1152[A^8+81B^8-84A^6B^2+756A^2B^6+630A^4B^4]-123904[A^7B-27AB^7-21A^5B^3+63A^3B^5]-1$$
$$p(A,B)=4A^2+12B^2$$

4. CONCLUSION

In this paper, we have presented sets of infinitely many non-zero distinct integer solutions to the sextic equation with five unknowns given by $2(x+y)(x^3-y^3)=61(z^2-w^2)p^4$.

As Diophantine equations are rich in variety due to their definition. One may attempt to find integer solutions to higher degree Diophantine equation with multiple variables.

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