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On The Cubic Equation with Five Unknowns $x^3+y^3=84(z+w)p^2$

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ABSTRACT

The cubic Equation $x^3+y^3=84(z+w)p^2$ is analyzed for its patterns of non – zero integral solutions. Five patterns of solutions are illustrated. A few properties among the solutions are presented.

Keywords: Cubic Equation with Five Unknowns and Integral solutions.

1. INTRODUCTION

Integral solutions for the homogeneous (or) non homogeneous Diophantine cubic equation is an interesting concept as it can be seen from [1, 2, 3]. In [4-9, 12, 13], a few special cases of cubic Diophantine equation with 3and 4 unknowns are studied. In [10, 11], cubic equations with 5 unknowns are studied for their integral solutions. In this communication, we present the integral solutions of an interesting cubic equation with 5 unknowns $x^3+y^3=84(z+w)p^2$. A few remarkable relations between the solutions are presented.

2. NOTATION USED

$t_{m,n}$ - Polygonal number of rank n with size m.

3. METHOD OF ANALYSIS

The cubic Diophantine equation with five unknowns to be solved is given by,

$$x^3+y^3=84(z+w)p^2 \quad (1)$$

The substitution of the linear transformation

$$x=u+v,\ y=u-v,\ z=u+p,\ w=u-p,\ u\neq v\neq 0 \quad (2)$$

in (1) leads to

$$84p^2=u^2+3v^2 \quad (3)$$

(3) is solved through different approaches and the different patterns of solutions of (1) obtained are presented below.

3.1 PATTERN: 1

Assume $p=a^2+3b^2$ (4)

Write as $84=(9+i\sqrt{3})(9-i\sqrt{3})$ (5)

Substituting (4) & (5) in (1) and employing the method of factorization, we have

$$(u+i\sqrt{3}v)(u-i\sqrt{3}v)=(9+i\sqrt{3})(9-i\sqrt{3})(a+i\sqrt{3}b)(a+i\sqrt{3}b)$$

Consider

$$u+i\sqrt{3}v=(9+i\sqrt{3})(a+i\sqrt{3}b)^2$$

$$u+i\sqrt{3}v=(9a^2-6ab-27b^2)+i\sqrt{3}(a^2+18ab-3b^2) \quad (6)$$

Equating real & imaginary parts

$$u=9a^2-6ab-27b^2$$

$$v=a^2+18ab-3b^2$$

Sub u, v & p in (2), we have

$$x=u+v=10a^2+12ab-30b^2$$

$$y=u-v=8a^2-24ab-24b^2$$

$$z=u+p=10a^2-6ab-24b^2$$

$$w=u-p=8a^2-6ab-30b^2$$

PROPERTIES

- $y(a,b)-z(a,b)-t_{6,a}\equiv 0(\mathrm{mod}\,19)$
- $2x(a,1)+y(a,1)-t_{58,a}\equiv 24(\mathrm{mod}\,27)$
- $z(a,1)-w(a,1)-t_{6,a}=a+6$
- $w(a,1)-x(a,1)+t_{6,a}\equiv 1(\mathrm{mod}\,17)$

3.2 PATTERN: 2

Write (3) as

$$u^2-81p^2=3(p^2-v^2) \quad (7)$$

Write (7) in the form of ratio as

$$\frac{u+9p}{p+v}=\frac{3(p-v)}{u-9p}=\frac{\alpha}{\beta},\beta\neq 0 \quad (8)$$

Which is equivalent to the system of double equations

$$\beta u-\alpha v+p(9\beta-\alpha)=0 \quad (9)$$

$$\alpha u+3v\beta-3p(\beta+3\alpha)=0 \quad (10)$$

Solving (5) & (6) by method of cross multiplication we've,

$$p = 3\beta^2 + \alpha^2$$
$$u = 9\alpha^2 + 6\alpha\beta - 27\beta^2 \qquad (11)$$
$$v = 3\beta^2 - \alpha^2 + 18\alpha\beta$$

Substituting (11) in (2), the integer solutions of (1) are given by,

$$x(\alpha,\beta) = 8\alpha^2 - 24\beta^2 + 24\alpha\beta$$
$$y(\alpha,\beta) = 10\alpha^2 - 30\beta^2 - 12\alpha\beta$$
$$z(\alpha,\beta) = 10\alpha^2 - 24\beta^2 + 6\alpha\beta$$
$$w(\alpha,\beta) = 8\alpha^2 - 30\beta^2 + 6\alpha\beta$$

PROPERTIES

- $x(\alpha,1) + 2y(\alpha,1) - t_{26,\alpha} \equiv 4 \pmod{11}$
- $z(\alpha,1) - w(\alpha,1) - t_{6,\alpha} = \alpha + 6$
- $z(\alpha,1) - x(\alpha,1) - t_{6,\alpha} \equiv 0 \pmod{17}$
- $2w(\alpha,1) - y(\alpha,1) + t_{6,\alpha} \equiv 0 \pmod{17}$

3.3 PATTERN: 3

Write (8) as

$$\frac{u+9p}{3(p+v)} = \frac{p-v}{u-9p} = \frac{\alpha}{\beta}, \beta \neq 0 \qquad (12)$$

which is equivalent to the system of double equations

$$\beta u - 3\alpha v + p(9\beta - 3\alpha) = 0 \qquad (13)$$
$$\alpha u + \beta v - p(\beta + 9\alpha) = 0 \qquad (14)$$

Solving (13) & (14) by method of cross multiplication we've,

$$p = \beta^2 + 3\alpha^2$$
$$u = 27\alpha^2 - 9\beta^2 + 6\alpha\beta \qquad (15)$$
$$v = \beta^2 - 3\alpha^2 + 18\alpha\beta$$

Substituting (15) in (2), the integer solutions of (1) are given by,

$$x(\alpha,\beta) = 24\alpha^2 - 8\beta^2 + 24\alpha\beta$$
$$y(\alpha,\beta) = 30\alpha^2 - 10\beta^2 - 12\alpha\beta$$
$$z(\alpha,\beta) = 30\alpha^2 - 8\beta^2 + 6\alpha\beta$$
$$w(\alpha,\beta) = 24\alpha^2 - 10\beta^2 + 6\alpha\beta$$

PROPERTIES

- $z(\alpha,1) - x(\alpha,1) + t_{14,\alpha} \equiv 0 \pmod{13}$
- $w(\alpha,1) - y(\alpha,1) + t_{10,\alpha} + t_{6,\alpha} \equiv 0 \pmod{14}$
- $z(\alpha,1) - w(\alpha,1) - t_{14,\alpha} \equiv 2 \pmod{5}$
- $y(\alpha,1) + 2z(\alpha,1) - t_{50,\alpha} - t_{40,\alpha} \equiv 26 \pmod{88}$

3.4 PATTERN: 4

Write (8) as

$$\frac{u+9p}{p-v} = \frac{3(p+v)}{u-9p} = \frac{\alpha}{\beta}, \beta \neq 0 \qquad (16)$$

which is equivalent to the system of double equations

$$\beta u + \alpha v + p(9\beta - \alpha) = 0 \qquad (17)$$
$$\alpha u - 3\beta v - p(3\beta + 9\alpha) = 0 \qquad (18)$$

Solving (17) & (18) by method of cross multiplication we've,

$$p = -3\beta^2 - \alpha^2$$
$$u = 9\alpha^2 - 27\beta^2 + 6\alpha\beta \qquad (19)$$
$$v = \alpha^2 - 3\beta^2 - 18\alpha\beta$$

Substituting (19) in (2), the integer solutions of (1) are given by,

$$x(\alpha,\beta) = 10\alpha^2 - 30\beta^2 - 12\alpha\beta$$
$$y(\alpha,\beta) = 8\alpha^2 - 24\beta^2 + 24\alpha\beta$$
$$z(\alpha,\beta) = 8\alpha^2 - 30\beta^2 + 6\alpha\beta$$
$$w(\alpha,\beta) = 10\alpha^2 - 24\beta^2 + 6\alpha\beta$$

PROPERTIES

- $3w(\alpha,1) - z(\alpha,1) - 6 = 6\alpha^2$ is a nasty number
- $z(\alpha,1) - x(\alpha,1) + t_{6,\alpha} \equiv 0 \pmod{17}$
- $y(\alpha,1) - w(\alpha,1) + t_{6,\alpha} \equiv 0 \pmod{17}$
- $4z(\alpha,1) - y(\alpha,1) - t_{50,\alpha} \equiv 6 \pmod{23}$

3.5 PATTERN: 5

Write (8) as

$$\frac{u+9p}{3(p-v)} = \frac{p+v}{u-9p} = \frac{\alpha}{\beta}, \beta \neq 0 \qquad (20)$$

which is equivalent to the system of double equations

$$\beta u + 3\alpha v + p(9\beta - 3\alpha) = 0 \qquad (21)$$
$$\alpha u - \beta v - p(\beta + 9\alpha) = 0 \qquad (22)$$

Solving (21) & (22) by method of cross multiplication, we've

$$p = -3\alpha^2 - \beta^2$$
$$u = 27\alpha^2 - 9\beta^2 + 6\alpha\beta \qquad (23)$$
$$v = 3\alpha^2 - \beta^2 - 18\alpha\beta$$

Substituting (23) in (2), the integer solutions of (1) are given by,

$$x(\alpha,\beta) = 30\alpha^2 - 10\beta^2 - 12\alpha\beta$$
$$y(\alpha,\beta) = 24\alpha^2 - 8\beta^2 + 24\alpha\beta$$
$$z(\alpha,\beta) = 24\alpha^2 - 10\beta^2 + 6\alpha\beta$$
$$w(\alpha,\beta) = 30\alpha^2 - 8\beta^2 + 6\alpha\beta$$

PROPERTIES

- $w(\alpha,1) - z(\alpha,1) - t_{14,\alpha} \equiv 2 \pmod{5}$
- $x(\alpha,1) - z(\alpha,1) + t_{14,\alpha} \equiv 0 \pmod{13}$
- $w(\alpha,1) - y(\alpha,1) + t_{10,\alpha} + t_{6,\alpha} \equiv 0 \pmod{14}$
- $2x(\alpha,1) + y(\alpha,1) - t_{74,\alpha} \equiv 7 \pmod{35}$

4. CONCLUSION
To conclude, one may search for other patterns of solutions and their corresponding properties.

REFERENCES
[1] L. E. Dickson, *History of Theory of numbers*, vol. 2, Chelsea Publishing Company, New York, 1952.

[2] L. J. Mordell, *Diophantine Equations*, Academic Press, London, 1969.

[3] R.D. Carmichael, *The Theory of numbers and Diophantine Analysis*, New York, Dover, 1959.

[4] M. A. Gopalan and S. Premalatha, "Integral solutions of $(x+y)(xy+w^2)=2(k^2+1)z^3$ ", Bulletin of Pure and Applied Sciences, Vol. 28E(No. 2), pp 197-202, 2009.

[5] M. A. Gopalan and V. Pandichelvi, "*Remarkable solutions on the cubic equation with four unknowns* $x^3+y^3+z^3=28(x+y+z)w^2$ *"*, Antarctica J. of Math., Vol. 4 , No. 4, Pp 393-401, 2010.

[6] M. A. Gopalan and B. Sivakami, "*Integral solutions of homogeneous cubic equation with four unknowns* $x^3+y^3+z^3=3xyz+2(x+y)w^3$ *"*, Impact. J.Sci.Tech, Vol.4, No.3, Pp 53-60, 2010.

[7] M. A. Gopalan and S. Premalatha, "*On the cubic Diophantine equation with four unknowns* $(x-y)(xy-w^2)=2(n^2+2n)z^3$ *"*, International Journal of Mathematical Sciences, Vol. 9, No. 1-2, Jan-June, Pp 171-175, 2010.

[8] M. A. Gopalan and J. Kalinga Rani, "*Integral solutions of* $x^3+y^3+(x+y)xy=z^3+w^3+(z+w)zw$ *"*, Bulletin of Pure and Applied Sciences, Vol. 29E (No. 1), Pp 169-173, 2010.

[9] M. A. Gopalan and S. Premalatha, "*Integral solutions of* $(x+y)(xy+w^2)=2(k+1)z^3$ *"* , The Global Journal of Applied Mathematics and Mathematical Sciences, Vol. 3, No. 1-2, Pp 51-55, 2010.

[10] M. A. Gopalan, S. Vidhyalakshmi and T. R. Usha Rani, "*On the cubic equation with five unknowns* $x^3+y^3=z^3+w^3+t^2(x+y)$ *"*, Indian Journal of Science, Vol. 1, No. 1, Pp 17-20, Nov 2012.

[11] M. A. Gopalan, S. Vidhyalakshmi and T. R. Usha Rani "*Integral solutions of the cubic equation with five unknowns* $x^3+y^3+u^3+v^3=3t^3$ *"*, IJAMA, Vol. 4(2), Pp 147-151, Dec 2012.

[12] M. A. Gopalan S. Vidhyalakshmi and E. Premalatha, "Cubic Diophantine Equation with three unknowns $(a+3)x^2-ay^2=27z^3$", Sch.J.Eng.Tech,2(5A), Pp 733-737, 2014.

[13] M. A. Gopalan, S. Vidhyalakshmi and E. Premalatha, "On the Cubic Equation with four unknowns $x^3+y^3=31(k^2+3s^2)zw^2$", IJSIMR, Vol.2, issue 11, pp 923-926, 2014.