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Number of Level Crossings of a Class of Random Algebraic Polynomial

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ABSTRACT

We know the expected number of times that a polynomial of degree n with independent normally distributed random real coefficients asymptotically crosses the line mx, when m is any real value such that $(m^2/n)\to 0$ as $n\to\infty$. The present paper shows that for m>exp (nf), where f is any function of n such that $f(n)\to\infty$, this expected number of crossings reduces to only one.

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1. Introduction

Consider the algebraic polynomial

$$P(x)=\sum_{i=0}^{n=1} a_t x^i \quad (1.1)$$

where $a_0,a_1,a_2,\ldots..a_{n-1}$ is a sequence of independent, normally distributed random variables with mathematical exception μ and variance unity. The set of equations y=P(x) represents a family of curves in the xy-plane, Kac [7], shows that for μ=0 the number of times that this family crosses the line x-axis, on a average is (2/π) log n. Later Ibragimov and Maslova [5] & [6] obtained the same average number of crossings when they considered the case of coefficients belongings to the domain of attraction of the normal law with mean μ=0. They also showed that when μ=0 the number of crossings reduces by half.

Farahmand [2], [3] & [4] studied the number of times that this family of curves crosses the level K=0(√n) (crosses with line y=K) for μ=0 and showed that these numbers decreases as K increases. He also showed that even in this case for μthe number of crossings reduces by half. Denote by N_m (a,b)= N (a,b) the number of times that this family crosses the line y=mx where m is any constant independent of x and let EN (a,b) be its expectation. For m-0(√n) an asymptotic value for EN(-∞,∞) was obtained by Farahamand, with which the reader will be assumed to be familiar. As noted in the latter there is a sizeable number of crossings even when the line tends to be perpendicular to the x axis, that is for m (=0 (√n)→∞ as n→∞. In this work we study the case when m is very large compared with n, and show that the number of crossings of this family of curves with such a line reduces to one. We prove.

Theorem-If the coefficients of P(x) in (1.1) are independent normally distributed random variables with mean zero and variance unity, then for any constant m such that $|m|>\exp\,(\mathrm{nf})$, where f is any function of n such that f(n) tends to infinity as n tends to infinity, the mathematical expectation of the number of real roots of the equation P(x)= mx is asymptotic to one.

2. Proof of the Theorem

First we find a lower estimates for EN (-∞,∞). Let m>exp (n,f), then since for $|x|<1$ the polynomial P(x) is convergent, with probability one, for x=1/2, say and n sufficiently large.

$$
\begin{aligned}
&P(x)=mx<P(1/2)-(1/2)\exp(nf)<0\\
&\textit{and}\ \text{also for } x=-1/2\\
&P(x)=-mx>P(-1/2)+(1/2)\exp(nf)>0
\end{aligned}
$$

Therefore, by the intermediate value theorem, there exists at least one real root for the function P(x)-mx in (-1/2, ½). Similarly, if m<-exp (nf) we can show that the function P(x)-mx takes on the opposite sign at x=1/2 and x=-1/2, therefore, there exists at least one real root. Hence EN $(-\infty,\infty)$ ≥1 and we only have to show that the upper limit is one as well. We also note that both aj and –aj (i=0,1,2,….n-1) have standard normal distribution hence changing x to –x leaves the distribution of the coefficients invariant, thus EN (-∞,0)= EN (-∞,0). So we only have to consider the interval (0,∞). In by using the expected number of level crossings in the work of Cramer and Lead better [1] and using the Kac-Rice [7] formula for the equation P(x)-mx=0 is found.

$$
\begin{aligned}
EN(a,b)=\int_a^b &\Big[(\Delta^{1/2}/\pi a)\exp\{(-\alpha m^2+2m^2\gamma x-\beta m^2x^2)/2\Delta\Big]\\
&+\left(\sqrt{2/\mathrm{n}}\right)m(\gamma x-\alpha)\big|\alpha^{-a/2}\exp(m_2x^2/2\alpha)erf\\
&\left\{\left|\mathrm{m}(\gamma\mathrm{x}-\alpha)\right|(2\alpha\Delta)^{-1/2}\right\}dx\\
&=\int_a^b I_1(x)dx+\int_a^b I_2(x)dx \qquad \text{say} \qquad (2.1)
\end{aligned}
$$

where

$$
\begin{aligned}
&\alpha=\sum_{t=\theta}^{n=1}x^{2i},\ \beta=\sum_{t=\theta}^{n=1}i^2x^{2i-2}\\
&\gamma=\sum_{t=1}^{n=1}ix^{2i-1},\ \Delta=\alpha\beta=\gamma^2 \qquad (2.2)
\end{aligned}
$$

and

$$erf(x)=\int_0^x \exp(-y^2)\,dy$$

First, we show $\int_0^1 I_1(x)$ dx tends to zero as n→∞. Let a be constant independent of x in the interval (0,1). For $0\le x\le 1-n^{-\alpha}$ and sufficiently large n we have

$$
\begin{aligned}
\gamma&=\{(n-1)x^{2n+1}-nx^{2n-1}+x\}(1-x^2)^{-2}\\
&=x(1-x^{2n})(1-x^2)^{-2}+0\{n^{1+\alpha}\exp(-2n^{1-\alpha})\}
\end{aligned}
$$

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and

$$\text{erf} = \int_0^x \exp(-y^2)dy$$

First we show that $\int_0^1 I_1(x)dx$ tends to zero as n→∞. Let a be a constant independent of x in the interval (0,1). For $0 \le x \le 1-n^{-a}$ and all sufficiently large n we have

$$\gamma = \{(n-1)x^{2n+1} - nx^{2n-1} + x\}(1-x^2)^{-2}$$
$$= x(1-x^{2n})(1-x^2)^{-2} + 0\{n^{1+\alpha}\exp(-2n^{1-\alpha})\}(2.3)$$

and

$$\beta = (1+x^2)(1-x^{2n})(1-x2)^{-3} + 0\{n^{2+a}\exp(-2n^{1-a}) \quad (2.4)$$

From (2.3) and (2.4) we can obtain

$$\Delta = (1-x^{2n})(1-x^2)^{-4} + 0\{n^{2+2n}\exp(-2n^{1-a}) \quad (2.5)$$

Now we choose $a=1-\{\log\log(n)^{10}\}/\log n$.

Then since, for all sufficiently large n, $n^{2+a}\exp(-2n^{1-a}) = n^{2+2n}\exp(-2\log(n)^{10}\} = n^{-18+a} \to 0.$

All the error terms that appear in the formulas (2.3) to (2.5) will tend to zero. Hence from (2.1), (2.3), (2.4), (2.5) and since for all x

$$(1-x^2)^{2/2} - x^2(1-x^2) + x^2(1+x^2)/2 > 1/5(2.6)$$

we have

$$\int_0^{1-n^{-a}} I_1(x)dx = \int_0^{1-n^{-a}} \left(\Delta^{1/2}/w\alpha\right)\exp\left[-m^2\{(1-x^2)/(1-x^{2n})\}\right]$$
$$\text{x } \{(1-x^2)^2/2 - x2(1-x^2)\}$$
$$+ x^2(1+x^2)/2\{(1+0\{n^{2+n}\exp(-2n^{1-a})\}dx$$

$$\le (1/\pi)\int_0^{1-n^{-a}} (1-x^2)^{-1}\exp\{-m^2(1-x^2)/5\}dx$$

$$\le (1/\pi)\exp(-m^2/2n^a)\int_0^{1-n^{-a}}(1-x^2)^{-1}dx$$
$$\le (1/2\pi)\exp\{-(m^2/2n)\log(n)^{10}\}\log\{n^{-a}(1-n^{-a})\}$$
$$\le (a/2\pi)\log n\exp\{-(m^2/2n)\log(n)^{10}\}.$$
$$\le (a/2\pi)\exp\{\log\log n\exp - (m^2/2n)\log\ (n)^{10}\}$$

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Now we note that since m>exp (n/logn) the term m^2/n tends to infinity much faster than log log n as $n\to\infty$, hence from (2.6) we can obtain

$$\int_0^{1-n^{-a}} I_1(x)dx \to 0 \; as \; n \to \infty. \qquad (2.7)$$

To show that $\int_0^{1-n^{-a}} I_1(x)dx \to 0 \; as \; n \to \infty.$ we first prove that $\left(\alpha - 2\gamma x + \beta x^2\right)/\Delta$ is positive for $1-n^{-a} \le x \le 1.$ For all sufficiently large n from (2.2) we have

$$\begin{aligned}
&\alpha - 2\gamma x + \beta x^2 \ge \beta x^2 - 2\gamma x \\
&\ge \{n^2 x^{2n}(1-x^2)2 - 2nx^{2n+2}(1-x^2)\} \\
&+ x^2(1+x^2)(1-x^{2n})\}(1-x^2) - 3 - n(n+1) \\
&\ge n^3\{\log(n)^{10}\}^{-3} - 2n^2 > n^2 \qquad (2.8)
\end{aligned}$$

since $(1-x^2)^3<(2n^{-a})^3$ and $2nx^{2n+2}\,(1-x^2)\to$ as $n\to\infty$. Hence from (2.8) and since $\Delta<n^4$ we have

$$(\alpha - 2\gamma x + \beta x^2)/\Delta > n^{-2} \qquad (2.9)$$

So from (2.9) and since from Farhamand[4] we have $(\Delta^2/a) < (2n-1)^{1/2}(1-x)^{-1/2}$, we have

$$\begin{aligned}
\int_0^{1-n^{-a}} I_1(x)dx &< (2n-1)^{1/2}\exp(-m^2/n^2)\int_0^{1-n^{-a}}(1-x)^{-1/2}dx \\
&< 3n^{(1-a)}/2\exp(-m^2/n^2) \qquad (2.10)
\end{aligned}$$

Which tends to zero as $n\to\infty$

In order to find $\int_0^{1-n^{-a}} I_1(x)dx$ we let $y=1/x$, and divide the interval $0\le y\le 1$ into three subintervals (0,b), (b,1-1/nd) and (1-1/nd,1) where d={(3/8) log log $(n)^{1/3}$ and b=$(m^{-2}$ n d log $n)^{1/(4n-8)}$,. We show that in these three subintervals $\int_0^{1-n^{-a}} I_1(x)dx = \int_0^1 y^{-2}\, I_1\,(1/y)\,dy = \log\{(1+b)/(1-b)\} \to 0 \text{ as } n\to\infty. \qquad (2.11)$

For b $\le y \le 1-1/nd$ *from* (2.2) we have

$$\alpha - 2\gamma/\gamma + \beta/\gamma^2 = 1 + \sum_{i=2}^{n} y^{-2i} + \sum_{i=2}^{n} y^{-2i}(i^2-2i) > n^3/4 \qquad (2.12)$$

and

$$\begin{aligned}
\Delta &= \{1-h^2(y)\}(1-y^{2n})2/y^{4n-8}(1-y^2)4 \le n^2/b^{4n-8}(1-y^2)^2 \\
&< 3n^4 d^2/b^{4n-8} \qquad (2.13)
\end{aligned}$$

where

$$h(y) = ny^{n-1}(1-y^2)/(1-y^{2n})$$

Hence from (2.12) and (2.13) we can write

$$\int_0^{1-1nd} y^{-2} I_1(1/y)dy \le \exp(-m^2 b^{4n-8}/12nd^2) \int_0^{1-1nd} (1-y^{-2})dy$$
$$\le (1/2)\log\,(2\mathrm{nd})\exp\,(\text{-}\log \mathrm{n}/12\,\mathrm{d}) \qquad (2.14)$$

which tends to zero as n tends to infinity. Finally, as for (2.10), we obtain

$$\int_{1-1/nd}^{1} y^{-2} I_1(1/y)dy < (2n-1)^{1/2} \int_{1-1/nd}^{1} y^{-2} I_1(1/y)dy < (2n-1)^{1/2}dy$$
$$< 2(2/\mathrm{d})^{1/2} \to 0 \; as \; \mathrm{n} \to \infty. \qquad (2.15)$$

Hence from (2.7), (2.10), (2.11), (2.14) and (2.15) we have

$$\int_0^{\infty} I_1(x)dx \to 0 \; as \; \mathrm{n} \to \infty. \qquad (2.16)$$

In what follows we will find an upper estimate for $\int_0^{\infty} I_2(x)dx$. From (2.2) we have

$$\left|m(\gamma x-\alpha)\alpha^{-3/2}\right| = \left|m\{(2x^2-1)(1-x^{2n})\right|$$
$$\text{-}\,\mathrm{nx}^{2\mathrm{n}}(1\text{-}\mathrm{x}^2)\}/(1\text{-}\mathrm{x}^2)^{1/2}(1\text{-}\mathrm{x}^{2\mathrm{n}})^{3/2} \qquad (2.17)$$

Now for $0 \le x \le \sqrt{3/2}$ for all $\mathrm{n} \ge 22$

nx^{2n} (1-x2)<n(3/4)^n<1/n

and

$$\left|(2x^2-1)(1-x^{2n})\right| \le \left|(2x^2-1)\right| < 1/n$$

only for (1-1/n)/2 <x^2<(1+1/n)/2. Let $\xi = \{(1-1/n)/2\}^{1/2}$ $and\ \xi' = \{(1+1/\mathrm{n})/2\}^{1/2}$ then from (2.17) and for all sufficiently large n we have

$$\int_{\xi}^{\xi'} I2(x) < (4|m|/n\sqrt{n})\exp(-m^2/2n)$$

which tends to zero as n tends to infinity. On the other hand, let (x) dx indicate the integral over $\alpha < (1-1/n)/2 \le x \le (1+1/n)/2 < b\ excluding\ (1-1/n)/2 \le x \le (1+1/n)/2.$ Let u=mx$\alpha^{-1/2}$, since da/dx=2y and erf (x) << $\sqrt{\pi/2}$ from (2.17) we have

$$\int_0^1 I_2(x) \leq (2n)^{-1/2} \int_0^{m/\sqrt{n}} \exp(-u^2/2)du < 1/2 \qquad (2.18)$$

To prove that

$$\int_0^\infty I_1(x)dx \to 0 \text{ as } n \to \infty, \textit{ from } (2.2) \text{ for } y = 1/x \text{ we have.}$$

$$(\gamma x - \alpha)/\alpha^{1/2}\Delta^{1/2} = \gamma^{2n-1}(1-y^2)^{3/2}\{(n-1-ny-y^{2n-1})(1-y^2)^2\}$$
$$- y^{2n-4}\}\{1-h^2(y)\}^{-1/2}(1-y^{2n})^{-1/2} \qquad (2.19)$$

and

$$x^2/\alpha = y^{2n-4}(1-y^2)/(1-y^{2n}) \qquad (2.20)$$

First we let

$$0 \leq y \leq (mn^2)^{-1/(2n-1)} \text{ then } y \to 0 \text{ as } n \to \infty, \text{ from } (2.19) \text{ we have}$$
$$(\gamma x - \alpha)/\alpha^{1/2}\Delta^{1/2} < 2ny^{2n-1} \qquad (2.21)$$

Let u= $mx\alpha^{-1/2}$ *and* $\lambda = (mn^2)^{-1/(2n-3)}$, then from (2.1), (2.20) and (2.21) we obtain

$$\int_0^\lambda y^{-2} I_2(1/y)\,dy < \text{erf}(1/n)\int_0^\infty \exp(-u^2/2) \to 0 \text{ as } n \to \infty. \qquad (2.22)$$

Since erf (1/n)→0 for all sufficiently large n. On the other hand for $\lambda \leq y \leq 1$ *we* let $u = mx\alpha^{-1/2}$, since

$$y^{2n-4}(1-y^2)/(1-y^{2n}) > 1/(2mn^2)$$
from (2.20) we have

$$\int_0^1 y^{-2} I_2(1/y)\,dy < \int_{m/2n2}^{m/\sqrt{n}} \exp(-u^2/2)du$$
$$\leq (m/\sqrt{n} - m^2/2n^2)\exp(-m^2/4n^4) \to 0 \text{ as } n \to \infty. \qquad (2.23)$$

Hence from (2.22) and (2.23) we have

$$\int_1^\infty I_2(x)\,dx \to 0 \text{ as } n \to \infty. \qquad (2.24)$$

Finally from (2.7), (2.10), (2.16), (2.18) and (2.24) we obtain

$$EN(0,\infty) \leq 1/2$$
$$\text{and since } EN(-\infty,\infty) = 2EN(0,\infty)$$

we have proof of the theorem.

3. Remark and Open Problem

The asymptotic number of crossings of the polynomial P(x) with line mx decreases as m=0(√n) increases. In this paper we proved that when $|m \geq \exp(nf)|$ the number of crossings reduces to one. The behaviour of the number of crossings between these two lines is not known. A subsequent study could consider the case when (m^2/n) tends to any non zero constant as n tends to infinity and as a guessed target $EN(-\infty,\infty) \sim (1/\pi)\log n$, which is half the number of crossings when m=0 seems reasonable. (Knowing a rough value for $EN(-\infty,\infty)$ is useful in order to sufficient upper and lower bounds for $EN(-\infty,\infty)$ leading to an asymptotic formula). Indeed, the behaviour of $EN(-\infty,\infty)$ for other values of m is also interesting, but it will involve more analysis especially for the $\int_{-\infty}^{\infty} \mathrm{I}_2(x)\, dx$ part of $EN(-\infty,\infty)$.

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Competing Interests Statement

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Consent for publication

Author declares that he consented for the publication of this research work.

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