1. Introduction

Consider the algebraic polynomial

$$P(x) = \sum_{i=0}^{n-1} a_i x^i$$  \hspace{1cm} (1.1)

where $a_0, a_1, a_2, \ldots, a_{n-1}$ is a sequence of independent, normally distributed random variables with mathematical exception $\mu$ and variance unity. The set of equations $y = P(x)$ represents a family of curves in the $xy$-plane, Kac [7], shows that for $\mu = 0$ the number of times that this family crosses the line $x$-axis, on an average is $(2/\pi) \log n$. Later Ibragimov and Maslova [5] & [6] obtained the same average number of crossings when they considered the case of coefficients belonging to the domain of attraction of the normal law with mean $\mu = 0$. They also showed that when $\mu = 0$ the number of crossings reduces by half.

Farahmand [2], [3] & [4] studied the number of times that this family of curves crosses the level $K = 0(\sqrt{n})$ (crosses with line $y = K$) for $\mu = 0$ and showed that these numbers decreases as $K$ increases. He also showed that even in this case for $\mu$ the number of crossings reduces by half. Denote by $N_m(a,b) = N(a,b)$ the number of times that this family crosses the line $y = mx$ where $m$ is any constant independent of $x$ and let $EN(a,b)$ be its expectation. For $m = 0(\sqrt{n})$ an asymptotic value for $EN(-\infty, \infty)$ was obtained by Farahmand, with which the reader will be assumed to be familiar. As noted in the latter there is a sizeable number of crossings even when the line tends to be perpendicular to the $x$ axis, that is for $m = 0(\sqrt{n}) \rightarrow \infty$ as $n \rightarrow \infty$. In this work we study the case when $m$ is very large compared with $n$, and show that the number of crossings of this family of curves with such a line reduces to one. We prove.

Theorem-If the coefficients of $P(x)$ in (1.1) are independent normally distributed random variables with mean zero and variance unity, then for any constant $m$ such that $|m| > \exp(nf)$, where $f$ is any function of $n$ such that $f(n)$ tends to infinity as $n$ tends to infinity, the mathematical expectation of the number of real roots of the equation $P(x) = mx$ is asymptotic to one.
2. Proof of the Theorem

First we find a lower estimates for $\text{EN}(-\infty, \infty)$. Let $m>\exp (n,f)$, then since for $|x| < 1$ the polynomial $P(x)$ is convergent, with probability one, for $x=1/2$, say and $n$ sufficiently large.

$$P(x) = mx < P(1/2) - (1/2) \exp (nf) < 0$$

and also for $x = -1/2$

$$P(x) = -mx > P(-1/2) + (1/2) \exp (nf) > 0$$

Therefore, by the intermediate value theorem, there exists at least one real root for the function $P(x)-mx$ in $(-1/2, 1/2)$. Similarly, if $m<\exp (nf)$ we can show that the function $P(x)-mx$ takes on the opposite sign at $x=1/2$ and $x=-1/2$, therefore, there exists at least one real root. Hence $\text{EN}(-\infty, \infty) \geq 1$ and we only have to show that the upper limit is one as well. We also note that both $a_j$ and $-a_j$ ($i=0,1,2,\ldots,n-1$) have standard normal distribution hence changing $x$ to $-x$ leaves the distribution of the coefficients invariant, thus $\text{EN}(-\infty,0) = \text{EN}(-\infty,0)$. So we only have to consider the interval $(0,\infty)$. In by using the expected number of level crossings in the work of Cramer and Lead better [1] and using the Kac-Rice [7] formula for the equation $P(x)-mx=0$ is found.

$$\text{EN}(a,b) = \int_a^b \left[ (\Delta^{1/2}/\pi a) \exp \left( (-am^2 + 2m^2 \chi - \beta m^2 x^2) / 2\Delta \right) ight]$$

$$+ \left( \frac{\sqrt{2m}}{\pi} \right) m (\gamma - \alpha) \alpha^{-1/2} \exp (m_2 x^2 / 2\alpha) \text{erf}$$

$$\left\{ m(\gamma - \alpha)(2\alpha \Delta)^{-1/2} \right\} dx$$

$$= \int_a^b I_1(x)dx + \int_a^b I_2(x)dx \quad \text{say} \quad (2.1)$$

where

$$\alpha = \sum_{i=0}^{n-1} x^{2i}, \beta = \sum_{i=0}^{n-1} i^2 x^{2i-2}$$

$$\gamma = \sum_{i=1}^{n-1} ix^{2i-1}, \Delta = 2\beta \gamma$$

(2.2)

and

$$\text{erf} (x) = \int_0^x \exp(-y^2) \ dy$$

First, we show $\int_0^1 I_1(x)dx$ tends to zero as $n\to \infty$. Let a be constant independent of $x$ in the interval $(0,1)$. For

$$0 \leq x \leq 1 - n^{-\alpha} \ \text{and sufficiently large} \ n \ \text{we have}$$

$$\gamma = \left\{(n-1)x^{2n+1} - nx^{2n-1} + x(1-x^2)^{-2} \right\}$$

$$= x(1-x^2)(1-x^2)^{-2} + 0(n^{1+\alpha} \exp(-2n^{1-\alpha}))$$

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and

\[
erf = \int_0^x \exp(-y^2) dy
\]

First we show that \( \int_0^1 I(x) dx \) tends to zero as \( n \to \infty \). Let \( a \) be a constant independent of \( x \) in the interval \((0,1)\). For \( 0 \leq x \leq 1 - n^{-a} \) and all sufficiently large \( n \) we have

\[
\gamma = \left( n - 1 \right) x^{2n+1} - nx^{2n} + x \left( 1 - x^2 \right) ^{-2}
= x \left( 1 - x^2 \right)^{2n} \left( 1 - x^2 \right)^{-2} + 0 \left( n^{1-a} \exp(-2n^{1-a}) \right) \quad (2.3)
\]

and

\[
\beta = (1 + x^2) \left( 1 - x^{2n} \right) \left( 1 - x^2 \right)^{-3} + 0 \left( n^{2-a} \exp(-2n^{1-a}) \right) \quad (2.4)
\]

From (2.3) and (2.4) we can obtain

\[
\Delta = (1 - x^{2n}) \left( 1 - x^2 \right)^{-4} + 0 \left( n^{2+2a} \exp(-2n^{1-a}) \right) \quad (2.5)
\]

Now we choose \( a = 1 - \{ \log \log (n)^{10} \}/\log n \).

Then since, for all sufficiently large \( n \),

\[
n^{2-a} \exp(-2n^{1-a}) = n^{2+2a} \exp(-2\log(n)^{10}) = n^{-18+a} \to 0.
\]

All the error terms that appear in the formulas (2.3) to (2.5) will tend to zero. Hence from (2.1), (2.3), (2.4), (2.5) and since for all \( x \)

\[
(1 - x^2)^{2/3} - x^2 (1 - x^2) + x^2 (1 + x^2)/2 > 1/5 \quad (2.6)
\]

we have

\[
I(x) dx = \int_0^{1 - n^{-a}} \left( \Delta \right)^{1/2} / \omega \alpha \exp \left\{ - m^2 \left( (1 - x^2)/(1 - x^{2n}) \right) \right\} dx
\]

\[
\int_0^{1 - n^{-a}} x \left( 1 - x^2 \right)^2 / 2 - x2(1 - x^2) \right\}
+ x^2(1 + x^2)/2 \left\{ 1 + 0 \left( n^{2+a} \exp (-2n^{1-a}) \right) \right\} dx
\]

\[
\leq \left( 1/\pi \right) \int_0^{1 - n^{-a}} \exp \left\{ - m^2 \left( (1 - x^2)/5 \right) \right\} dx
\]

\[
\leq \left( 1/\pi \right) \exp \left( - m^2 / 2n^a \right) \int_0^{1 - n^{-a}} \left( 1 - x^2 \right)^{-1} dx
\]

\[
\leq (1/2\pi) \exp \left\{ - (m^2/2n) \log \left( n^{10} \right) \log \left( n^{-a} (1 - n^{-a}) \right) \right\}
\]

\[
\leq (a/2\pi) \log n \exp \left\{ - (m^2/2n) \log \left( n^{10} \right) \right\}.
\]

\[
\leq (a/2\pi) \exp \{ \log \log n \exp - (m^2/2n) \log \left( n^{10} \right) \}
\]
Now we note that since \( m > \exp(n \log n) \) the term \( m^2/n \) tends to infinity much faster than \( \log \log n \) as \( n \to \infty \), hence from (2.6) we can obtain

\[
\int_0^{1-n^{-\alpha}} I_1(x) \, dx \to 0 \quad \text{as} \quad n \to \infty. \tag{2.7}
\]

To show that \( \int_0^{1-n^{-\alpha}} I_1(x) \, dx \to 0 \quad \text{as} \quad n \to \infty. \) we first prove that \( (\alpha - 2\gamma x + \beta x^2) / \Delta \) is positive for \( 1 - n^{-\alpha} \leq x \leq 1. \) For all sufficiently large \( n \) from (2.2) we have

\[
\alpha - 2\gamma x + \beta x^2 \geq \beta x^2 - 2\gamma x \\
\geq \left\{ n^2 x^{2n} (1 - x^2) 2 - 2nx^{2n+2} (1 - x^2) \right\} \\
+ x^2 (1 + x^2) (1 - x^{2n}) (1 - x^2) - 3n(n+1) \\
\geq n^3 \{ \log(n)^{1/3} \} - 2n^2 > n^2 \tag{2.8}
\]

since \( 1 - x^2 \leq (2n^{-3}) \) and \( 2nx^{2n+2} (1 - x^2) \to \) as \( n \to \infty. \) Hence from (2.8) and since \( \Delta < n^4 \) we have

\[
(\alpha - 2\gamma x + \beta x^2) / \Delta > n^{-2} \tag{2.9}
\]

So from (2.9) and since from Farhamand[4] we have \( (\Delta^2 / a) < (2n-1)^{1/2} (1 - x)^{-1/2} \), we have

\[
\int_0^{1-n^{-\alpha}} I_1(x) \, dx < (2n-1)^{1/2} \exp(-m^2/n^2) \int_0^{1-n^{-\alpha}} (1-x)^{-1/2} \, dx \\
< 3n^{(1-a)/2} \exp(-m^2/n^2) \tag{2.10}
\]

Which tends to zero as \( n \to \infty. \)

In order to find \( \int_0^{1-n^{-\alpha}} I_1(x) \, dx \) we let \( y = 1/x, \) and divide the interval \( 0 \leq y \leq 1 \) into three subintervals \( (0,b), (b,1-1/nd), \) and \( (1-1/nd,1) \) where \( d=\{(3/8) \log \log (n)^{(1/3)} \) and \( b=\left(m^2 n d \log n\right)^{1/(4n-8)} \). We show that in these three subintervals

\[
\int_0^{1-n^{-\alpha}} I_1(x) \, dx = \int_0^1 y^{-2} I_1(1/y) \, dy = \log \{(1 + b)/(1 - b)\} \to 0 \quad \text{as} \quad n \to \infty. \tag{2.11}
\]

For \( b \leq y \leq 1 - 1/nd \) from (2.2) we have

\[
\alpha - 2\gamma / \gamma + \beta / \gamma^2 = 1 + \sum_{i=2}^n y^{-2i} + \sum_{i=2}^n y^{-2i}(i^2 - 2i) > n^3 / 4 \tag{2.12}
\]

and

\[
\Delta = \left\{ 1 - h^2(y) \right\} (1 - y^2n) 2 / y^{4n-8} (1 - y^2) 4 \leq n^3 / b^{4n-8} (1 - y^2)^{3} \\
< 3n^4 \, d^3 / b^{4n-8} \tag{2.13}
\]
where
\[ h(y) = ny^{n-1}(1 - y^2)/(1 - y^{2n}) \]

Hence from (2.12) and (2.13) we can write
\[
\int_{0}^{1-1/nd} y^{-2}I_1(1/y)dy \leq \exp(-m^2 b^{4n-8}/12nd^2) \int_{0}^{1-1/nd} (1 - y^{-2})dy
\]
\[
\leq (1/2) \log (2nd) \exp(-\log n/12d)
\]
which tends to zero as \( n \) tends to infinity. Finally, as for (2.10), we obtain
\[
\int_{1-1/nd}^{1} y^{-2}I_1(1/y)dy < (2n-1)^{1/2} \int_{1-1/nd}^{1} y^{-2}I_1(1/y)dy < (2n-1)^{1/2}dy
\]
\[
< 2(2/d)^{1/2} \rightarrow 0 \text{ as } n \rightarrow \infty.
\]
Hence from (2.7), (2.10), (2.11), (2.14) and (2.15) we have
\[
\int_{0}^{\infty} I_1(x)dx \rightarrow 0 \text{ as } n \rightarrow \infty.
\]

In what follows we will find an upper estimate for \( \int I_2(x)dx \). From (2.2) we have
\[
|m(\alpha x - \alpha)\alpha^{-3/2}| = |m((2x^2 - 1)(1 - x^{2n})|
\]
\[
- nx^{2n}(1 - x^2)/(1 - x^2)^{1/2}(1 - x^{2n})^{3/2}
\]
(2.17)

Now for \( 0 \leq x \leq \sqrt{3/2} \) for all \( n \geq 22 \)
\[
x^{2n}(1-x^2)<n(3/4)^n<1/n
\]
and
\[
|(2x^2 - 1)(1 - x^{2n})| \leq |2x^2 - 1| < 1/n
\]
only for \((1-1/n)/2 < x < (1+1/n)/2\). Let \( \xi = (1-1/n)/2 \) and \( \xi' = (1+1/n)/2 \) then from (2.17) and for all sufficiently large \( n \) we have
\[
\int_{\xi}^{\xi'} I_2(x) < (4|m|/n\sqrt{n}) \exp(-m^2/2n)
\]
which tends to zero as \( n \) tends to infinity. On the other hand, let \( \int x \ dx \) indicate the integral over \( \alpha < (1-1/n)/2 \leq x \leq (1+1/n)/2 < b \) excluding \( (1-1/n)/2 \leq x \leq (1+1/n)/2 \). Let \( u = mx^n \alpha^{1/2} \), since \( da/dx = 2y \) and \( \text{erf}(x) << \sqrt{\pi}/2 \) from (2.17) we have

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\[
\int_0^1 I_2(x) \leq (2n)^{-1/2} m/\sqrt{n} \int_0^{m/\sqrt{n}} \exp(-u^2/2)du < 1/2 \tag{2.18}
\]

To prove that
\[
\int_0^\infty I_1(x)dx \rightarrow 0 \text{ as } n \rightarrow \infty, \text{ from (2.2) for } y=1/x \text{ we have.}
\]

\[
(y - \alpha) / \alpha^{1/2} \Delta^{1/2} = y^{2n-1} (1 - y^2)^{3/2} \left[ (n - 1 - ny - y^{2n-1}) (1 - y^2)^2 \right]
- y^{2n-4} \{1 - h^2(y)\}^{-1/2} (1 - y^{2n})^{-1/2} \tag{2.19}
\]

and
\[
x^2 / \alpha = y^{2n-4} (1 - y^2) / (1 - y^{2n}) \tag{2.20}
\]

First we let
\[
0 \leq y \leq (mn^-2)^{-1/2(n-1)} \text{ then } y \rightarrow 0 \text{ as } n \rightarrow \infty, \text{ from (2.19) we have}
\]

\[
(y - \alpha) / \alpha^{1/2} \Delta^{1/2} < 2ny^{2n-1} \tag{2.21}
\]

Let \( u = \int_0^y y^2I_2 (1/y) dy < \exp(-u^2/2) \rightarrow 0 \text{ as } n \rightarrow \infty. \tag{2.22} \]

Since \( \exp(-u^2/2) \rightarrow 0 \text{ for all sufficiently large } n. \) On the other hand for \( \lambda \leq y \leq 1 \text{ we let } u = mx \alpha^{1/2}, \text{ since}
\]

\[
y^{2n-4} (1 - y^2) / (1 - y^{2n}) > 1/(2mn^2)
\]

from (2.20) we have
\[
\int_0^1 y^2I_2 (1/y) dy < \int_0^{m/\sqrt{n}} \exp(-u^2/2)du
\]

\[
\leq (m/\sqrt{n-m^2/2n^2}) \exp(-m^2/4n^2) \rightarrow 0 \text{ as } n \rightarrow \infty. \tag{2.23}
\]

Hence from (2.22) and (2.23) we have
\[
\int_1^\infty I_2(x) dx \rightarrow 0 \text{ as } n \rightarrow \infty. \tag{2.24}
\]

Finally from (2.7), (2.10), (2.16), (2.18) and (2.24) we obtain
\[
EN (0, \infty) \leq 1/2
\]

and since \( EN (-\infty, \infty) = 2EN (0, \infty) \)

we have proof of the theorem.
3. Remark and Open Problem

The asymptotic number of crossings of the polynomial $P(x)$ with line $mx$ decreases as $m=0(\sqrt{n})$ increases. In this paper we proved that when $|m| \geq \exp(n^{f})$ the number of crossings reduces to one. The behaviour of the number of crossings between these two lines is not known. A subsequent study could consider the case when $(m^2/n)$ tends to any non zero constant as $n$ tends to infinity and as a guessed target $\mathbb{E}(\mathbb{N}(\infty, \infty) \sim (1/\pi) \log n$, which is half the number of crossings when $m=0$ seems reasonable. (Knowing a rough value for $\mathbb{E}(\mathbb{N}(\infty, \infty)$ is useful in order to sufficient upper and lower bounds for $\mathbb{E}(\mathbb{N}(\infty, \infty)$ leading to an asymptotic formula). Indeed, the behaviour of $\mathbb{E}(\mathbb{N}(\infty, \infty)$ for other values of $m$ is also interesting, but it will involve more analysis especially for the

$$\int_{-\infty}^{\infty} I_2(x) \, dx$$

part of $\mathbb{E}(\mathbb{N}(\infty, \infty)$.

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