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Solving Partial Integro Differential Equations using Modified Differential Transform Method

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ABSTRACT

In this paper, we introduced the modified differential transform which is a modified version of a two-dimensional differential transform method. First, the properties of the modified differential transform method (MDTM) are presented. After this, by using the idea modified differential transform method we will find an analytical-numerical solution of linear partial integro-differential equations (PIDE) with convolution kernel which occur naturally in various fields of science and engineering. In some cases, the exact solution may be achieved. The efficiency and reliability of this method are illustrated by some examples.

Keywords: Modified differential transform method, Partial integro-differential equation, Maclaurin's Series, Differential transform method, Integro differential equation.

1. Introduction

The theory and applications of PIDE's play an important role in the various fields of science and engineering. The analysis of such PIDE can be found in [5-8]. In the last few years, Jyoti Thorwe and Sachin Bhalekar [1] used Laplace transform method and Mohand M. Abdelrahim Mehgob and Tarig M. Elzaki [3] used Elzaki transform method to solve linear partial integro-differential equations (PIDE) with convolution kernel. They converted the PIDE to an ordinary differential equation (ODE).

Solving this ODE and applying inverse transform they obtain the exact solution of the problem. Ranjit Dhunde and G. L. Waghmare [2] used double Laplace transform and Mohand M. Abdelrahim Mehgob [4] applied double Elzaki transform to solve PIDE. They converted PIDE to an algebraic equation and Appling inverse transform they obtained exact solution.

Differential Transform Method is a semi-analytical numerical technique which depends on Taylor's series for the solution of differential and integral equations. The concept of differential transform method was first introduced by Zhou [9] who solved linear and non-linear initial value problems in electric circuit analysis.

Recently, The various types of differential equations, integro differential equations, and Volterra integral equations solved by using two-dimensional DTM [12-19]. The modified differential transform method is a modified version of two-dimensional DTM, and it will take less computational time and effort to solve PIDEs.

In this paper, we will find an analytical-numerical solution of linear partial integro differential equations (PIDE) with convolution kernel using a modified differential transform method.

2. PRELIMINARIES

2.1 Partial Integro Differential Equation (PIDE)

The general form of a linear PIDE with convolution kernel (see [1-4])

$$\sum_{i=1}^{m} a_i\left(\frac{\partial^i u}{\partial x^i}\right)+\sum_{i=1}^{n} b_i\left(\frac{\partial^i u}{\partial t^i}\right)+cu(x,t)+\sum_{i=1}^{r} d_i\int_0^t K_i(t-y)\left(\frac{\partial^i u}{\partial x^i}\right)dy+f(x,t)=0 \qquad (1)$$

Where, a_i , b_i , c and d_i are constants or the functions of x alone. And $f(x,t)$, $K_i(t-y)$ are known functions.

2.2 Modified Differential Transform Method (MDTM)

Taylor's series expansion of the function $u(x,t)$ with respect to specific variable $t=t_0$ is,

$$U(x,h)=\frac{1}{h!}\left\{\frac{\partial^h}{\partial t^h}u(x,t)\right\}_{t=t_0} \quad ; \quad h\geq 0 \qquad (2)$$

Where, $u(x,t)$ the original is function and $U(x,h)$ is transformed function.

Therefore, the inverse modified differential transform of a function $U(x,h)$ is define by,

$$u(x,t)=\sum_{h=0}^{\infty}U(x,h)(t-t_0)^h \quad ; \quad h\geq 0 \qquad (3)$$

Note: $t_0=0$

$$U(x,h)=\frac{1}{h!}\left\{\frac{\partial^h}{\partial t^h}u(x,t)\right\}_{t=0} \quad ; \quad h\geq 0 \qquad (4)$$

$$\therefore\ u(x,t)=\sum_{h=0}^{\infty}U(x,h)t^h \quad ; \quad h\geq 0$$

We summarized modified differential transforms of some standard functions in the following table [22, 23].

Table 1: MDTM w. r. t t of some standard functions

Original Function $u(x,t)$	**Transformed Function $U(x,h)$**
$u(x)v(t)$	$u(x)V(h)$
$x^m t^n$	$x^m\delta(h-n)$
t^n	$\delta(h-n)$
x^m	$x^m\delta(h)$
e^{ax+bt}	$e^{ax}\frac{b^h}{h!}$
$x^m sinat$	$\frac{x^m}{h!}sin\left(\frac{h\pi}{2}\right)$
$x^m cosat$	$\frac{x^m}{h!}cos\left(\frac{h\pi}{2}\right)$

We summarized some fundamental properties of modified differential transforms in the following table [22, 23].

Theorem 1 [22,23]: If modified differential transform of $f(x,t)$, $u(x,t)$, $g(t)$ and $v(t)$ are $F(x,h), U(x,h), G(h)$ and $V(h)$ respectively then,

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$$a)\ If\ \ u(x,t)=\frac{\partial^m y(x,t)}{\partial x^m}\quad then\quad U(x,h)=\frac{d^m}{dx^m}Y(x,h)$$

$$b)\ If\ \ u(x,t)=\frac{\partial^n y(x,t)}{\partial t^n}\quad then\quad U(x,h)=(h+1)(h+2)\ldots.(h+n)Y(x,h+n)$$

$$c)\ If\ \ u(x,t)=\frac{\partial^{m+n} y(x,t)}{\partial x^m \partial t^n}\quad then\ \ U(x,h)=(h+1)..(h+n)\frac{d^m}{dx^m}Y(x,h+n)$$

$$d)\ If\ \ f(x,t)=u(x,t)v(x,t)\quad then\quad F(x,h)=\sum_{l=0}^{h}U(x,l)V(x,h-l)$$

$$e)\ If\ f(x,t)=\int_0^t u(x,y)dy\quad then\ \ F(x,h)=\begin{cases}\dfrac{U(x,h-1)}{h} & ;\ \ if\ h\geq 1\\ 0; & if\ \ h=0\end{cases}$$

$$f)\ If\ f(x,t)=g(t)u(x,t)\quad then\ \ F(x,h)=\sum_{l=0}^{h}G(l)U(x,h-l)$$

$$g)\ If\ f(x,t)=\int_0^t g(y)\,u(x,y)dy\quad then\ F(x,h)=\begin{cases}\displaystyle\sum_{l=0}^{h-1}\frac{G(l)U(x,h-1-l)}{h} & ;\ \ if\ h\geq 1\\ 0; & if\ \ h=0\end{cases}$$

$$h)\ If\ f(x,t)=g(t)\int_0^t u(x,y)\,dy\quad then\ \ F(k,h)=\begin{cases}\displaystyle\sum_{l=0}^{h-1}\frac{G(l)U(x,h-l-1)}{h-l}; & h\geq 1\\ 0; & h=0\end{cases}$$

2.3 Solving PIDE Using MDTM

Consider PIDE,

$$\sum_{i=0}^{m}a_i\left(\frac{\partial^i u}{\partial x^i}\right)+\sum_{i=0}^{n}b_i\left(\frac{\partial^i u}{\partial t^i}\right)+cu(x,t)+\sum_{i=0}^{r}d_i\int_0^t K_i(t-y)\left(\frac{\partial^i u}{\partial x^i}\right)dy+f(x,t)=0$$

Appling MDTM, we get

$$\therefore\sum_{i=0}^{m}a_i\frac{d^i}{dx^i}U(x,h)\ +\sum_{i=1}^{l}b_iU(x,h+j)\left(\prod_{j=1}^{i}(h+j)\right)+\ cU(k,h)+D\left\{\sum_{i=1}^{r}d_i\int_0^t K_i(t-y)\left(\frac{\partial^i u}{\partial x^i}\right)dy\right\}$$
$$+F(x,h)=0$$

After expanding kernel function $k_i(t-y)$ in the form as $\emptyset(t)\psi(y)$. Then apply MDTM properties.

Now we introduce the new property of MDTM in the following theorem for solving a linear PIDE with convolution kernel problems.

Theorem 2: If modified differential transform of $f(x,t),\ u(x,t),\ g(t)$ and $V(h)$ are $F(x,h), U(x,h), G(h)$ and $V(h)$ are respectively and

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If $f(x,t) = g(t)\int_0^t v(y)\,u(x,y)\,dy$ then $F(x,h) = \begin{cases} \sum_{l=1}^{h}\sum_{s=0}^{l-1}\frac{1}{l}\,G(h-l)V(s)U(x,l-s-1) & ; \quad h \geq 0 \\ 0 & ; \quad h = 0 \end{cases}$

Proof: Define, $w(x,t) = \int_0^t v(y)u(x,y)\,dy$

$$\therefore f(x,t) = g(t)\,w(x,t)$$

Consider,

$$\therefore h!\,F(x,h) = \left\{\frac{\partial^h}{\partial t^h}g(t)w(x,t)\right\}_{t=0}$$

Applying Leibnitz's theorem,

$$h!\,F(x,h) = \left\{\sum_{l=0}^{h}\binom{h}{l}\frac{\partial^l w(x,t)}{\partial t^l}\frac{\partial^{h-l} g(t)}{\partial t^{h-l}}\right\}_{t=0}$$

$$h!\,F(x,h) = \sum_{l=0}^{h}\frac{h!}{(h-l)!\,l!}\,l!\,W(x,l)(h-l)!\,G(h-l)$$

$$\therefore F(x,h) = \sum_{l=0}^{h}W(x,l)G(h-l) \quad (5)$$

As $w(x,t) = \int_0^t v(y)u(x,y)\,dy$

Using theorem 1 (15),

$$W(x,l) = \begin{cases} \sum_{s=0}^{l-1}\frac{1}{l}\,V(s)\,U(x,l-1-s) & ; \quad l \geq 1 \\ 0 & ; \quad l = 0 \end{cases} \quad (6)$$

$$\therefore F(x,h) = \sum_{l=0}^{h}G(h-l)\begin{cases} \sum_{s=0}^{l-1}\frac{1}{l}\,V(s)\,U(x,l-1-s) & ; \; l \geq 1 \\ 0; & l = 0 \end{cases}$$

As $l = 0$; $F(x,h) = 0$

$\therefore$ Replace $l = 0$ to $l = 1$ in the first summation,

$$\therefore F(x,h) = \begin{cases} \sum_{l=1}^{h}\sum_{s=0}^{l-1}\frac{1}{l}\,G(h-l)V(s)U(x,l-s-1) & ; \; l-1 \geq 0 \;\; i.e.\, h \geq 1 \\ 0; \quad l = 0 \quad i.e. \; h = 0 \end{cases}$$

3. Applications

Example 1: Consider PIDE [1],

$$u_t - u_{xx} + u + \int_0^t e^{(t-y)} u(x,y)dy = (x^2+1)e^t - 2$$

With initial conditions, $u(x,0) = x^2$, $u_t(x,0) = 1$

Solution: Given,

$$u_t - u_{xx} + u + \int_0^t e^{(t-y)} u(x,y)dy = (x^2+1)e^t - 2$$

$$\therefore u_t - u_{xx} + u + e^t \int_0^t e^{-y} u(x,y)dy = x^2e^t + e^t - 2 \quad (7)$$

with initial conditions,

$$u(x,0) = x^2 , \quad u_t(x,0) = 1 \quad (8)$$

Applying MDTM on both sides of (7) and (8),

$$\therefore U(x,h+1) = \frac{1}{(h+1)} \left\{ \begin{array}{c} \frac{d^2}{dx^2}U(x,h) - U(x,h) + \frac{x^2}{h!} + \frac{1}{h!} - 2\delta(h) \\ -\sum_{l=1}^{h}\sum_{s=0}^{l-1}\left(\frac{1}{l}\right)\left(\frac{1}{(h-l)!}\right)\left(\frac{(-1)^s}{s!}\right)U(x,l-s-1) \end{array} \right\} \quad (9)$$

and $U(x,0) = x^2, \quad U(x,1) = 1$ (10)

Put $h = 0,1,2,3,4, \ldots.$ in equation (9) and using (10),

If $h = 1 \ \therefore U(x,2) = 0$, If $h = 2 \ \therefore U(x,3) = 0$ & If $h = 3 \ \therefore U(x,4) = 0$

In general,

$\therefore \ h \geq 2 \quad \therefore U(x,h) = 0$

We know that,

$$\therefore u(x,t) = \sum_{h=0}^{\infty} U(x,h)t^h$$

$$= t + x^2$$

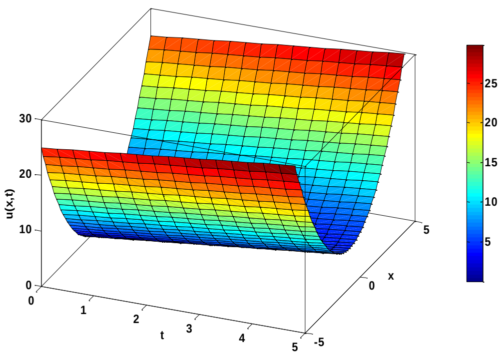

Fig.1 *Solution of* $u(x,t) = x^2 + t$

Example 2: Consider PIDE [1],

$$xu_x = u_{tt} + xsint + \int_0^t sin(t-y)u(x,y)dy$$

with initial conditions, $u(x,0) = 0,\ u_t(x,0) = x$

Solution: Given,

$$xu_x = u_{tt} + xsint + \int_0^t \sin(t-y)\, u(x,y)dy$$

$$u_{tt} = xu_x - xsint - sint\int_0^t cosy\, u(x,y)dy + cost\int_0^t siny\, u(x,y)dy \quad (11)$$

with initial conditions, $u(x,0) = 0,\ u_t(x,0) = x$ (12)

Applying MDTM on both sides of (11) and (12),

$$U(x,h+2) = \frac{1}{(h+1)(h+2)}\left\{\begin{array}{c} x\frac{d}{dx}U(x,h) - x\frac{sin\left(\frac{h\pi}{2}\right)}{h!} \\ -\sum_{l=1}^{h}\sum_{s=0}^{l-1}\left(\frac{1}{l}\right)\left(\frac{sin\left(\frac{(h-l)\pi}{2}\right)}{(h-l)!}\right)\left(\frac{cos\left(\frac{s\pi}{2}\right)}{s!}\right)U(x,l-s-1) \\ +\sum_{l=1}^{h}\sum_{s=0}^{l-1}\left(\frac{1}{l}\right)\left(\frac{cos\left(\frac{(h-l)\pi}{2}\right)}{(h-l)!}\right)\left(\frac{sin\left(\frac{s\pi}{2}\right)}{s!}\right)U(x,l-s-1) \end{array}\right\}$$

Using $SinACosB - CosASinB = Sin(A-B)$

$$U(x,h+2) = \frac{1}{(h+1)(h+2)}\left\{\begin{array}{c} x\frac{d}{dx}U(x,h) - x\frac{sin\left(\frac{h\pi}{2}\right)}{h!} \\ -\sum_{l=1}^{h}\sum_{s=0}^{l-1}\left(\frac{1}{l}\right)\left(\frac{sin\left(\frac{(h-l-s)\pi}{2}\right)}{(h-l)!\ \ s!}\right)U(x,l-s-1) \end{array}\right\} \quad (13)$$

and $U(x,0) = 0, \quad U(x,1) = x$ (14)

Put $h = 0,1,2,3,4, \ldots.$ in equation (13) and using (14),

If $h = 0 \quad \therefore U(x,2) = 0$

If $h = 1 \quad \therefore U(x,3) = 0$

If $h = 2 \quad \therefore U(x,4) = 0$

In general,

For all $h \geq 2 \ \therefore U(x,h) = 0$

We know that,

$$\therefore u(x,t) = \sum_{h=0}^{\infty} U(x,h)t^h$$

$$= xt$$

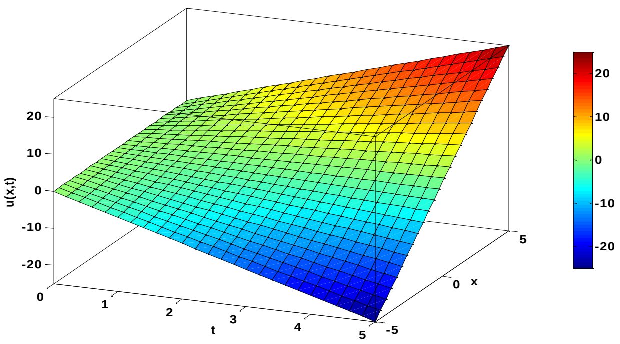

Fig.2 *Solution of* $u(x,t) = xt$

Example 3: Consider PIDE [1]

$$u_t + u_{ttt} - \int_0^t sinh(t-y)\, u_{xxx}(x,y)dy = 0$$

With initial condition $u(x,0) = 0,\ u_t(x,0) = x\ \ ,\ u_{tt}(x,0) = 0$

Solution: Given,

$$u_t + u_{ttt} - \int_0^t sinh(t-y)\, u_{xxx}(x,y)dy = 0$$

$$\therefore u_{ttt} = -u_t + sinht \int_0^t coshy\, u(x,y)dy - cosht \int_0^t sinhy\, u(x,y)dy \tag{15}$$

With initial conditions, $u(x,0) = 0\ ,u_t(x,0) = x\ \ ,u_{tt}(x,0) = 0$ (16)

Applying MDTM on both sides of (15) and (16),

$$\therefore U(x,h+3) = \frac{1}{(h+1)(h+2)(h+3)} \left\{ \begin{array}{c} -(h+1)U(x,h+1) \\ + \sum_{l=1}^{h}\sum_{s=0}^{l-1} \left(\frac{1}{l}\right)\left(\frac{1}{(2(h-l)+1)!}\right)\left(\frac{1}{(2s)!}\right)\frac{d^3}{dx^3}U(x,l-s-1) \\ -\sum_{l=1}^{h}\sum_{s=0}^{l-1} \left(\frac{1}{l}\right)\left(\frac{1}{(2(h-l))!}\right)\left(\frac{1}{(2s+1)!}\right)\frac{d^3}{dx^3}U(x,-1) \end{array} \right\} \tag{17}$$

and $U(x,0) = 0\ ,\quad U(x,1) = x,\quad U(x,2) = 0$ (18)

Put $h = 0,1,2,3,4,....$ in equation (17) and using (18), If $h = 0 \qquad \therefore U(x,3) = -\frac{x}{3!}$

If $h = 1 \qquad \therefore U(x,4) = 0$

If $h = 2 \qquad \therefore U(x,5) = \frac{x}{5!}$

If $h = 3 \qquad \therefore U(x,6) = 0$

If $h = 4 \qquad \therefore U(x,7) = -\frac{x}{7!}$

In general,

$$\therefore U(x,2m+1) = (-1)^m \frac{x}{(2m+1)!} \quad \forall\, m \quad and \quad U(x,2m) = 0; \forall\, m$$

We know that,

$$\therefore u(x,t) = \sum_{h=0}^{\infty} U(x,h)t^h$$

$$= x\left(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \frac{t^9}{9!} - \cdots\ldots\right)$$

$$= x\, sint$$

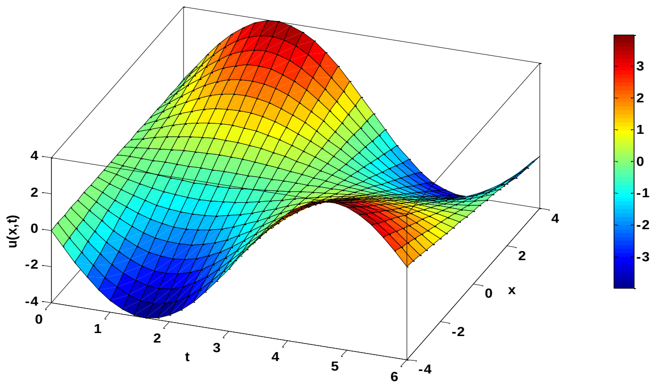

Fig.3 *Solution of* $u(x,t) = x\, sint$

4. Conclusion

PIDE's are used in the modelling various phenomenon in science and engineering. The modified differential transform method is successfully used to solve linear partial integro-differential equations (PIDE) with convolution kernel. We get an analytical-numerical solution. We concluded that to solve PIDEs by other traditional methods required initial as well as boundary conditions but MDTM method required only initial conditions to solve PIDEs. It is observed that MDTM takes less computational time and effort and it is a very powerful, efficient technique to solve PIDEs. In some cases, the exact solution may be achieved. We hope some other types of PIDE can be used in various fields modelling real-life phenomena.

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