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A New Approach of Newton's Method using Simpson's Rule

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ABSTRACT

In the proposed method, a new approach has introduced to modify Newton's iterative method to find the solution of nonlinear equations using local linear model and to interpret the model, Newton's theorem is used involving indefinite integral. Moreover, the indefinite integral is approximated by the Simpson's rule to derive the derivative of the function. In addition, the order of convergence is computed.

Keywords: Local linear model, Newton's method, Non-linear equations, Order of convergence and Simpson's rule.

1. INTRODUCTION

In numerical analysis, there are various iterative methods to find the solution of a nonlinear equation

$$f(x) = 0 \tag{1}$$

and Newton's method is one of the most powerful methods to find roots because it gives accurate result with fewer steps comparing with other methods. Though the method based on initial approximation, which plays an important role to find solution because if the initial approximation is not correctly chosen then we will get divergent result. But if the initial approximation is correctly chosen, then after some iterations, the process doubles the number of correct decimal places or significant digits at each iteration and thus very high accuracy obtained quickly [1, 5]. In this paper, we use a method to modify Newton's iterative formula, by approximating the derivative value by the indefinite integral and then evaluating the integral value by the Simpson's rule, which is based on [1], where they approximated the indefinite integral by the trapezoidal rule. Moreover, we discuss the order of convergence. Though the order of convergence is of first degree using Simpson's rule, which is not satisfactory but Weerakoon, S. and T. G. I. Fernando [5] found third order convergence using trapezoidal rule.

2. DESCRIPTION OF THE METHOD

Newton's quadratically convergent iterative formula is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \tag{2}$$

Where x_0 is the initial approximation which is sufficiently close to α, the root of the nonlinear equation (1). In [5], Weerakoon, S. and T. G. I. Fernando discussed the construction of Newton's method. At each step they constructed a local linear model of $f(x)$ at the point x_n and solved for the root x_{n+1} of the local model. In Newton's method, this local linear model is the tangent drawn to the function $f(x)$ at the current point x_n. Local linear model at x_n is

$$M_n(x) = f(x_n) + f'(x_n)(x - x_n). \tag{3}$$

In [3], Dennis, J. E. and R. B. Schnable, constructed the local linear model differently. From Newton's theorem,

$$f(x) = f(x_n) + \int_{x_n}^{x} f'(\lambda)d\lambda. \tag{4}$$

In Newton's method, the indefinite integral is approximated by the rectangle, i.e.,

$$\int_{x_n}^{x} f'(\lambda)\, d\lambda \approx f'(x_n)(x - x_n), \tag{5}$$

Which gives the result model (2).

Weerakoon, S. and T. G.I. Fernando [5], approximated the indefinite integral by the trapezoidal rule and they constructed the third order convergent iterative method

$$x_{n+1} = x_n - \frac{2f(x_n)}{[f'(x_n)+f'(x_{n+1}^*)]}, \; n = 0, 1, 2, \dots \dots \dots .. \tag{6}$$

Where
$$x_{n+1}^* = x_n - \frac{f(x_n)}{f'(x_n)}. \tag{7}$$

We use the same approach in [5] in the following to derive the formula, but we use Simpson's rule [2] to approximate the derivative function.

Using Simpson's rule [2] to approximate the following integral,

$$\int_{x_n}^{x} f(\lambda)\, d\lambda \approx \frac{(x-x_n)}{6}\left\{f(x_n) + 4f\left(\frac{x_n+x}{2}\right) + f(x)\right\} \tag{8}$$

and then for approximating derivative the above integral becomes,,

$$\int_{x_n}^{x} f'(\lambda)\, d\lambda \approx \frac{(x-x_n)}{6}\left\{f'(x_n) + 4f'\left(\frac{x_n+x}{2}\right) + f'(x)\right\} \tag{9}$$

Then the local linear model (3) becomes,

$$M_n(x) = f(x_n) + \frac{(x-x_n)}{6}\left\{f'(x_n) + 4f'\left(\frac{x_n+x}{2}\right) + f'(x)\right\} \tag{10}$$

We take the next iterative point as the root of the local model (10).

$$M_n(x_{n+1}) = 0$$

$$\Rightarrow \; f(x_n) + \frac{(x_{n+1}-x_n)}{6}\left\{f'(x_n) + 4f'\left(\frac{x_n+x_{n+1}}{2}\right) + f'(x_{n+1})\right\} = 0$$

$$\Rightarrow \; x_{n+1} = x_n - \frac{6f(x_n)}{\left[f'(x_n)+4f'\left(\frac{x_n+x_{n+1}}{2}\right)+f'(x_{n+1})\right]} \tag{11}$$

Let us assume $x_{n+1} = z$ and $y_n = \frac{x_n+x_{n+1}}{2}$ and substituting the values in equation (11), we find the three step iteration method which is given in the following algorithm.

2.1 Algorithm

For an initial approximation x_0, we can approximate solution x_{n+1}, by the three step iteration formula:

$$x_{n+1} = x_n - \frac{6f(x_n)}{[f'(x_n)+4f'(y_n)+f'(z_n)]} \tag{12}$$

Where
$$z_n = x_n - \frac{f(x_n)}{f'(x_n)} \tag{13}$$

and
$$y_n = \frac{x_n+z_n}{2} \tag{14}$$

for $n = 0, 1, 2, \dots \dots \dots \dots \dots \dots \dots$

3. CONVERGENCE ANALYSIS

We use the same approach in [4] to state the following theorem to show that the proposed method which is given by the equations (12), (13) and (14) has first order convergence.

3.1 Theorem: Let $\alpha \in I$ be a simple root of sufficiently differentiable function $f: I \subseteq R \rightarrow R$ for an open interval I. Then the three step iteration method (12), (13) and (14) has first order convergence.

Proof: Let α be a simple root of f. So $f(\alpha) = 0$. Since f is sufficiently differentiable. So $f(x_n)$ can be expanded in Taylor's series about α,

$$f(x_n) = f(\alpha) + f'(\alpha)(x_n - \alpha) + \frac{f''(\alpha)}{2!}(x_n - \alpha)^2 + \frac{f'''(\alpha)}{3!}(x_n - \alpha)^3 + \; \dots \dots \quad (15)$$

Let $e_n = x_n - \alpha$. Then equation (15) implies,

$$f(x_n) = 0 + f'(\alpha)e_n + \frac{f''(\alpha)}{2!}e_n^2 + \frac{f'''(\alpha)}{3!}e_n^3 + O(e_n^4) \quad (16)$$

Let $c_k = \frac{f^{(k)}(\alpha)}{k!\, f'(\alpha)}$ for $k = 2, 3, \dots \dots \dots \dots$ in equation (16),

$$f(x_n) = f'(\alpha)[e_n + c_2 e_n^2 + c_3 e_n^3 + O(e_n^4)] \quad (17)$$

Then differentiating equation (17) about e_n,

$$f'(x_n) = f'(\alpha)[1 + 2c_2 e_n + 3c_3 e_n^2 + O(e_n^3)] \quad (18)$$

In equation (13), we put the values of $f(x_n)$ and $f'(x_n)$,

$$z_n = \alpha + c_2 e_n^2 + (2c_3 - 2c_2^2)e_n^3 + O(e_n^4) \quad (19)$$

Then we put the value of x_n and z_n in equation (14), we get

$$y_n = \alpha + \frac{1}{2}e_n + \frac{1}{2}c_2 e_n^2 + (c_3 - c_2^2)e_n^3 + O(e_n^4) \quad (20)$$

Expanding $f(z_n)$ and $f(y_n)$ in a Taylor series about α and using $f(\alpha) = 0$,

$$f(z_n) = f'(\alpha)[c_2 e_n^2 + 2(c_3 - c_2^2)e_n^3 + O(e_n^4)] \quad (21)$$

and

$$f(y_n) = f'(\alpha)\left[\frac{1}{2}e_n + \frac{3}{4}c_2 e_n^2 + \frac{1}{2}(2c_3 - c_2^2)e_n^3 + O(e_n^4)\right] \quad (22)$$

Then differentiating $f(z_n)$ and $f(y_n)$ about e_n, we get

$$f'(z_n) = f'(\alpha)[2c_2 e_n + 6(c_3 - c_2^2)e_n^3 + O(e_n^3)] \quad (23)$$

and

$$f'(y_n) = f'(\alpha)\left[\frac{1}{2} + \frac{3}{2}c_2 e_n + \frac{3}{2}(2c_3 - c_2^2)e_n^2 + O(e_n^3)\right] \quad (24)$$

Adding the equations (18) and (23) with four times of equation (24),

$$f'(x_n) + 4f'(y_n) + f'(z_n) = f'(\alpha)[3 + 10c_2 e_n + (21c_3 - 12c_2^2)e_n^2 + O(e_n^3)] \quad (25)$$

By using equations (18) and (25) in the following expression,

$$\frac{6f(x_n)}{[f'(x_n) + 4f'(y_n) + f'(z_n)]} = 2e_n - \frac{14}{3}c_2 e_n^2 + \left(\frac{212c_2^2 - 108c_3}{9}\right)e_n^3 + O(e_n^4) \quad (26)$$

Now we put the value from equation (26) and x_n in equation (12),

$$x_{n+1} = \alpha - e_n + \frac{14}{3}c_2 e_n^2 - \left(\frac{212c_2^2 - 108c_3}{9}\right)e_n^3 + O(e_n^4) \quad (27)$$

Let $x_{n+1} = \alpha + e_{n+1}$, then equation (27) becomes,

$$e_{n+1} = -e_n + \frac{14}{3}c_2 e_n^2 - \left(\frac{212c_2^2 - 108c_3}{9}\right) e_n^3 + O(e_n^4) \quad (28)$$

Hence, it is proved that the iteration method has first order convergence.

4. CONCLUSION

In this paper, we have applied the Simpson's rule [3] to approximate the first derivative value of the function and we substituted the value in the well-known Newton's Iterative formula to modify it and then we introduced a three step iterative method. Moreover, we have computed the order of convergence for this method. This method has first order convergence where Newton's method is quadratically convergent. Hence, we have found that the convergence rate is slower comparing to the Newton's method and also comparing to the method in [5] where they have got third order convergence.

5. CONFLICT OF INTEREST

The author declares no conflict of interest.

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