Asian Journal of Applied Science and Technology (AJAST)
(Open Access Quarterly International Journal) Volume 1, Issue 9, Pages 305-313, 2017

# Stability Analysis of Dynamics of Variable Mass System

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*Article Received: 05 August 2017* *Article Accepted: 30 September 2017* *Article Published: 19 November 2017*

ABSTRACT

This paper addresses a stability analysis of a heavy point with variable mass. The dynamic equation of the variable mass is considered and different stability conditions in the sense of Lypunov were established for a given trajectory in the vertical plane. To justify the accuracy of the results obtained, a specific numerical example is provided and simulation results are displayed using MatLab.

Keywords: Asymptotic stability, Constraint, Nonlinear dynamic system, Stability in the sense of Lyapunov and Variable mass.

## 1. STATEMENT OF THE PROBLEM

Variable-mass systems are systems which have mass that do not remain constant with respect to time. It is known that in such systems, Newton's second law of motion cannot directly be applied. Instead, a body whose mass m varies with time can be described by Newton's second law by adding a term to account for the momentum carried by mass entering or leaving the system [1],[2], [3].

$$\boldsymbol{F} = \frac{d\boldsymbol{p}}{dt} = \frac{d(m\boldsymbol{v})}{dt} = m\frac{d\boldsymbol{v}}{dt} + \boldsymbol{v}\frac{dm}{dt} \ , \tag{1}$$

Variable mass systems have been the focus of a large number of problems in different areas like mechanics and mathematics by different scholars [4],[5].Consider a particle of mass m which is moving with velocity (v). Its linear momentum isp=mv. According to Newton's second law, the change of the linear momentum p in time t is determined by

$$\boldsymbol{F} = m\frac{d\boldsymbol{v}}{dt} = m\boldsymbol{a}. \tag{2}$$

Where F is the net force acting on the particle. If the particle has a constant mass, the above equation reduces to Newton's second law of motion described as:

$$\frac{d(m\boldsymbol{v})}{dt} = \boldsymbol{F} + \boldsymbol{u}\frac{dm}{dt}, \tag{3}$$

However, in practice we encounter the movement of a body with variable mass. This is the case for rockets, balloons, falling drops and ice crystals. The differential equation of translator motion of a rigid body whose mass m is dependent on time is of the form

$$\frac{d(m\boldsymbol{v})}{dt} = \boldsymbol{F} + \boldsymbol{u}\frac{dm}{dt}, \tag{3}$$

Where F is the resultant of all forces acting on the body and u is the velocity of the added mass before being joined to the body (dm/dt>0), or that of detracted mass after being separated from the body (dm/dt<0). If we do not consider the dependence of m on speed we [5] obtain

$$m\frac{d\boldsymbol{v}}{dt} = \boldsymbol{F} + (\boldsymbol{u} - \boldsymbol{v})\frac{dm}{dt}. \tag{4}$$

Unlike the case of linear systems, proving stability of equilibrium points of nonlinear systems is more complicated. A sufficient condition [6],[7],[8],[9],[10] is the existence of a Lyapunov function. Lyapunov's indirect method that investigates the local stability of the equilibrium is inconclusive when the linearized system has imaginary axis eigenvalues.

The stability analysis of a dynamic system with variable mass is, to the best of understanding of the researchers, is less common in the existing literature. Consequently, the main target of this article is to carry out stability analysis of a heavy point with variable mass constrained to move along a trajectory in the $xz$-plane, in the vertical direction based on the dynamic equation of a variable mass [5] developed by Mesheriskii I.V. given by equation (5) below. The paper established a sufficient stability condition for different given cases in the sense of Lyapunov direct method.

$$\begin{cases} \ddot{x} = \frac{\dot{m}}{m}(\mu - 1)\dot{x} - \frac{k_1}{m}\sqrt{\dot{x}^2 + \dot{z}^2}\dot{x} - \frac{k_2}{m}\sqrt{\dot{x}^2 + \dot{z}^2}\dot{z} \\ \ddot{z} = \frac{\dot{m}}{m}(\eta - 1)\dot{z} - \frac{k_1}{m}\sqrt{\dot{x}^2 + \dot{z}^2}\dot{z} + \frac{k_2}{m}\sqrt{\dot{x}^2 + \dot{z}^2}\dot{x} - g \end{cases}, \tag{5}$$

Where m = m (t) is the mass of the heavy point ($\dot{m}$<0), μ=μ(t),η=η(t)-the projection of the velocity of the varying mass point on the$x$, $z$ –coordinate plane respectively,

X=k_1 v^2,Z=k_2 v^2- normal and tangential components of the external force acting on the point mass (the impact of the environmental force acting on the point mass) respectively , g-gravitational acceleration, k_1=k_1 (t),k_2=k_2 (t)- coefficients of the impact force acting on the point mass along the $X$ and $Z$-components respectively, and v=√(x ̇^2+z ̇^2 )-velocity of the point mass in the vertical plane.

## 2. STABILITY ANALYSIS

Consider eq.(5).

$$\begin{cases} \ddot{x} = \frac{\dot{m}}{m}(\mu-1)\dot{x} - \frac{k_1}{m}\sqrt{\dot{x}^2+\dot{z}^2}\dot{x} - \frac{k_2}{m}\sqrt{\dot{x}^2+\dot{z}^2}\dot{z} \\ \ddot{z} = \frac{\dot{m}}{m}(\eta-1)\dot{z} - \frac{k_1}{m}\sqrt{\dot{x}^2+\dot{z}^2}\dot{z} + \frac{k_2}{m}\sqrt{\dot{x}^2+\dot{z}^2}\dot{x} - g \end{cases}. \quad (6)$$

Suppose that the point mass is constrained to move along a trajectory given by:

$$\begin{cases} x = \varphi(t) \\ z = \psi(t) \end{cases}. \quad (7)$$

Substituting the values of x and z given in eq.(7) into eq.(6) and considering ψ ̇,φ ̇≠0, we obtain a sufficient condition that makes the heavy point mass to move on the constraint trajectory. Solving for μ and η leads to the following system of equations.

$$\begin{cases} \mu = 1 + \frac{m}{\dot{m}}\left(\frac{\ddot{\varphi}}{\dot{\varphi}} + \frac{k_1\dot{\varphi}+k_2\dot{\psi}}{m\dot{\varphi}}\sqrt{\dot{\varphi}^2+\dot{\psi}^2}\right) \\ \eta = 1 + \frac{m}{\dot{m}}\left(\frac{\ddot{\psi}}{\dot{\psi}} + \frac{g}{\dot{\psi}} + \frac{k_1\dot{\psi}-k_2\dot{\varphi}}{m\dot{\psi}}\sqrt{\dot{\varphi}^2+\dot{\psi}^2}\right) \end{cases}. \quad (8)$$

Using the free variables μ, η and m, it is possible to establish conditions for stability of the motion along the given trajectory described by eq. (7).

In the following sections, stability conditions for the motion of the point mass along the trajectory given by eq. (7) relative to newly introduced position coordinates(x_1,x_3) and velocity coordinates (x_2,x_4)is discussed. Transforming eq.(8) to a systems of first order differential equations yields:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = \left(\frac{\ddot{\varphi}}{\dot{\varphi}} + \frac{1}{m\dot{\varphi}}\frac{k_2\dot{\psi}\dot{\varphi}^2+k_2\dot{\psi}^3}{\sqrt{\dot{\varphi}^2+\dot{\psi}^2}}\right)x_2 - \left(\frac{k_2\dot{\varphi}^2+k_2\dot{\psi}^2}{m\sqrt{\dot{\varphi}^2+\dot{\psi}^2}}\right)x_4 \\ \dot{x}_3 = x_4 \\ \dot{x}_4 = \frac{k_2}{m}\left(\frac{\dot{\varphi}^2+\dot{\psi}^2}{\sqrt{\dot{\varphi}^2+\dot{\psi}^2}}\right)x_2 + \left(\frac{\ddot{\psi}}{\dot{\psi}} + \frac{g}{\dot{\psi}} - \frac{(k_2\dot{\varphi}^3+k_2\dot{\varphi}\dot{\psi}^2)}{m\dot{\psi}\sqrt{\dot{\varphi}^2+\dot{\psi}^2}}\right)x_4 \end{cases}, \quad (9)$$

Asian Journal of Applied Science and Technology (AJAST)
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Where x_(2, ) x_4 are velocities and x_(1, ) x_3 are position coordinates.

It is assumed that the coefficients in eq. (9) are continuous and bounded for all t≥t_0[8],[11].

Consider the velocity equations in the system of eq. (9)

$$\begin{cases} \dot{x}_2 = \left(\frac{\ddot{\varphi}}{\dot{\varphi}} + \frac{1}{m\dot{\varphi}}\frac{k_2\dot{\psi}\dot{\varphi}^2+k_2\dot{\psi}^3}{\sqrt{\dot{\varphi}^2+\dot{\psi}^2}}\right)x_2 - \left(\frac{k_2\dot{\varphi}^2+k_2\dot{\psi}^2}{m\sqrt{\dot{\varphi}^2+\dot{\psi}^2}}\right)x_4 \\ \dot{x}_4 = \frac{k_2}{m}\left(\frac{\dot{\varphi}^2+\dot{\psi}^2}{\sqrt{\dot{\varphi}^2+\dot{\psi}^2}}\right)x_2 + \left(\frac{\ddot{\psi}}{\dot{\psi}} + \frac{g}{\dot{\psi}} - \frac{(k_2\dot{\varphi}^3+k_2\dot{\varphi}\dot{\psi}^2)}{m\dot{\psi}\sqrt{\dot{\varphi}^2+\dot{\psi}^2}}\right)x_4 \end{cases} . \tag{10}$$

To investigate the global stability of the system in eq.(10) let us consider a Lyapunov function candidate given by:

$$V(x_2,x_4) = {x_2}^2 + {x_4}^2 ,$$

Which is obviously positive definite function and is radially unbounded.

Now,

$$\frac{dV}{dt} = \frac{\partial V}{\partial x_2}\dot{x}_2 + \frac{\partial V}{\partial x_4}\dot{x}_4$$

$$= 2x_2\left[\left(\frac{\ddot{\varphi}}{\dot{\varphi}} + \frac{1}{m\dot{\varphi}}\frac{k_2\dot{\psi}\dot{\varphi}^2 + k_2\dot{\psi}^3}{\sqrt{\dot{\varphi}^2 + \dot{\psi}^2}}\right)x_2 - \left(\frac{k_2\dot{\varphi}^2 + k_2\dot{\psi}^2}{m\sqrt{\dot{\varphi}^2 + \dot{\psi}^2}}\right)x_4\right]$$

$$+ 2x_4\left[\frac{k_2}{m}\left(\frac{\dot{\varphi}^2 + \dot{\psi}^2}{\sqrt{\dot{\varphi}^2 + \dot{\psi}^2}}\right)x_2 + \left(\frac{\ddot{\psi}}{\dot{\psi}} + \frac{g}{\dot{\psi}} - \frac{(k_2\dot{\varphi}^3 + k_2\dot{\varphi}\dot{\psi}^2)}{m\dot{\psi}\sqrt{\dot{\varphi}^2 + \dot{\psi}^2}}\right)x_4\right]$$

$$= 2\left[\left(\frac{\ddot{\varphi}}{\dot{\varphi}} + \frac{1}{m\dot{\varphi}}\frac{k_2\dot{\psi}\dot{\varphi}^2 + k_2\dot{\psi}^3}{\sqrt{\dot{\varphi}^2 + \dot{\psi}^2}}\right){x_2}^2 - \frac{k_2}{m}\left(\frac{\dot{\varphi}^2 + \dot{\psi}^2}{\sqrt{\dot{\varphi}^2 + \dot{\psi}^2}}\right)x_2x_4\right]$$

$$+ 2\left[\frac{k_2}{m}\left(\frac{\dot{\varphi}^2 + \dot{\psi}^2}{\sqrt{\dot{\varphi}^2 + \dot{\psi}^2}}\right)x_2x_4 + \left(\frac{\ddot{\psi}}{\dot{\psi}} + \frac{g}{\dot{\psi}} - \frac{(k_2\dot{\varphi}^3 + k_2\dot{\varphi}\dot{\psi}^2)}{m\dot{\psi}\sqrt{\dot{\varphi}^2 + \dot{\psi}^2}}\right){x_4}^2\right]$$

$$= 2\left[\left(\frac{\ddot{\varphi}}{\dot{\varphi}}{x_2}^2 + \left(\frac{\ddot{\psi}}{\dot{\psi}} + \frac{g}{\dot{\psi}}\right){x_4}^2\right) - \frac{1}{m\sqrt{\dot{\varphi}^2+\dot{\psi}^2}}\left(\frac{k_2\dot{\varphi}^3+k_2\dot{\varphi}\dot{\psi}^2}{\dot{\psi}}{x_4}^2 + \frac{k_2\dot{\psi}\dot{\varphi}^2+k_2\dot{\psi}^3}{\dot{\varphi}}{x_2}^2\right)\right].$$

Substituting

$$R(t, x_2, x_4) \equiv \frac{(k_2\dot{\varphi}^3+k_2\dot{\varphi}\dot{\psi}^2)}{\dot{\psi}}{x_4}^2 + \frac{k_2\dot{\psi}\dot{\varphi}^2+k_2\dot{\psi}^3}{\dot{\varphi}}{x_2}^2,$$

We obtain

$$\frac{dV}{dt} = 2\left[\left(\frac{\ddot{\varphi}}{\dot{\varphi}}{x_2}^2 + \left(\frac{\ddot{\psi}}{\dot{\psi}} + \frac{g}{\dot{\psi}}\right){x_4}^2\right) - \frac{R(t,x_2,x_4)}{m\sqrt{\dot{\varphi}^2+\dot{\psi}^2}}\right].$$

Hence, for dV/dt to be negative in a certain region

$${x_2}^2 + {x_4}^2 \leq H, t \geq t_0,$$

The sufficient condition that needs to be satisfied is:

$$\begin{cases} \frac{\ddot{\varphi}}{\dot{\varphi}} < 0 \\ \frac{\ddot{\psi}}{\dot{\psi}} + \frac{g}{\dot{\psi}} < 0 \\ \frac{\dot{\varphi}^2}{\dot{\psi}} > -\dot{\psi} \\ \dot{\varphi} < -\frac{\dot{\psi}^2}{\dot{\varphi}} \end{cases}. \quad (11)$$

Consequently, the motion of the heavy point mass is asymptotically stable in its velocity provided that the conditions in eq.(11) are satisfied.

Moreover, it is possible to verify that when the conditions in eq.(11) are satisfied, the heavy point mass motion is asymptotically stable in terms of its position coordinates$(x_1, x_3)$.

***Example1*:** Let $x = \varphi(t)\ and\ z = a\varphi(t)$, where $a \neq 1\ and\ \dot{\varphi} \neq 0$

Substituting $x = \varphi(t)\ and\ z = a\varphi(t)$ in the stability conditions given in eq.(11), we obtain

$$\begin{cases} \frac{\ddot{\varphi}}{\dot{\varphi}} < 0, \\ \frac{\ddot{\varphi}}{\dot{\varphi}} + \frac{g}{a\dot{\varphi}} < 0, \\ k_2 > -k_2 a^2, \\ k_2 a^2 < -k_2, \text{for } a > 0. \end{cases}$$

### *2.1 The case when there is no Tangential Force acting on the mass* ( $k_2 = 0$).

In this case, eq. (10) is reduces to

$$\begin{cases} \dot{x}_2 = \frac{\ddot{\varphi}}{\dot{\varphi}} x_2 \\ \dot{x}_4 = \left(\frac{\ddot{\psi}}{\dot{\psi}} + \frac{g}{\dot{\psi}}\right) x_4 \end{cases}.$$

Suppose the Lyapunov function candidate is given by:

$$V = e^{m(t)}({x_2}^2 + {x_4}^2),$$

Where m(t) is the mass of the heavy point. Then,

$$\frac{dV}{dt} = \frac{\partial V}{\partial x_2}\dot{x}_2 + \frac{\partial V}{\partial x_4}\dot{x}_4$$

$$= e^m \left(\dot{m} + 2\frac{\ddot{\varphi}}{\dot{\varphi}}\right){x_2}^2 + e^m \left(\dot{m} + 2\frac{\ddot{\psi}}{\dot{\psi}} + 2\frac{g}{\dot{\psi}}\right){x_4}^2.$$

The condition that needs to be satisfied for asymptotic stability (dV/dt<0) is:.

$$\begin{cases} \dot{m} < -2\frac{\ddot{\varphi}}{\dot{\varphi}} \\ \dot{m} < -2\frac{\ddot{\psi}}{\dot{\psi}} - 2\frac{g}{\dot{\psi}} \end{cases}. \qquad (12)$$

***Example 2***: Let

$$x = \varphi = t \implies \dot{x} = \dot{\varphi} = 1 \implies \ddot{x} = \ddot{\varphi} = 0,$$

$$z = \psi = -10t \implies \dot{z} = \dot{\psi} = -10 \implies \ddot{z} = \ddot{\psi} = 0,$$

And

$$x = \varphi = t \implies \dot{x} = \dot{\varphi} = 1 \implies \ddot{x} = \ddot{\varphi} = 0,$$

$$z = \psi = -10t \Longrightarrow \dot{z} = \dot{\psi} = -10 \Longrightarrow \ddot{z} = \ddot{\psi} = 0,$$

Then based on eq.(12), the asymptotic stability conditions are

$$-2e^{-2t} < 0 \text{ and} -2(e^{-2t} + 1) < 0.$$

Indeed, let us claim that the system is stable for Lyapunov function given by:

$$V = e^{m(t)}({x_2}^2 + {x_4}^2).$$

Then

$$\frac{dV}{dt} = \frac{\partial V}{\partial x_2}\dot{x}_2 + \frac{\partial V}{\partial x_4}\dot{x}_4$$

$$= e^m\left(\dot{m} + 2\frac{\ddot{\varphi}}{\dot{\varphi}}\right){x_2}^2 + e^m(\dot{m} + 2\frac{\ddot{\psi}}{\dot{\psi}} + 2\frac{g}{\dot{\psi}})\,{x_4}^2.$$

Substituting the values of m,φ,ψ,g,m ̇,φ ̇,ψ ̇,φ ̈ and ψ ̈, from the given example we have:

$$\dot{V} = e^{e^{-2t}}(-2e^{-2t} + 0){x_2}^2 + e^{e^{-2t}}(-2e^{-2t} - 2){x_4}^2$$

$$= -2\left[\left(e^{-2t+e^{-2t}}\right){x_2}^2 + 2\left(e^{-2t+e^{-2t}} + e^{e^{-2t}}\right){x_4}^2\right] < 0.$$

That is, V ̇<0 for non-zero values of x_2,x_4.

Let us see the simulation result using MATLAB 2008B based on the given data in this example.Since,k_2=0, it follows that the system represented by eq.(9)reduces to:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = \dfrac{\ddot{\varphi}}{\dot{\varphi}}x_2 \\ \dot{x}_3 = x_4 \\ \dot{x}_4 = \left(\dfrac{\ddot{\psi}}{\dot{\psi}} + \dfrac{g}{\dot{\psi}}\right)x_4 \end{cases}.$$

Accordingly, the coefficient matrix A is

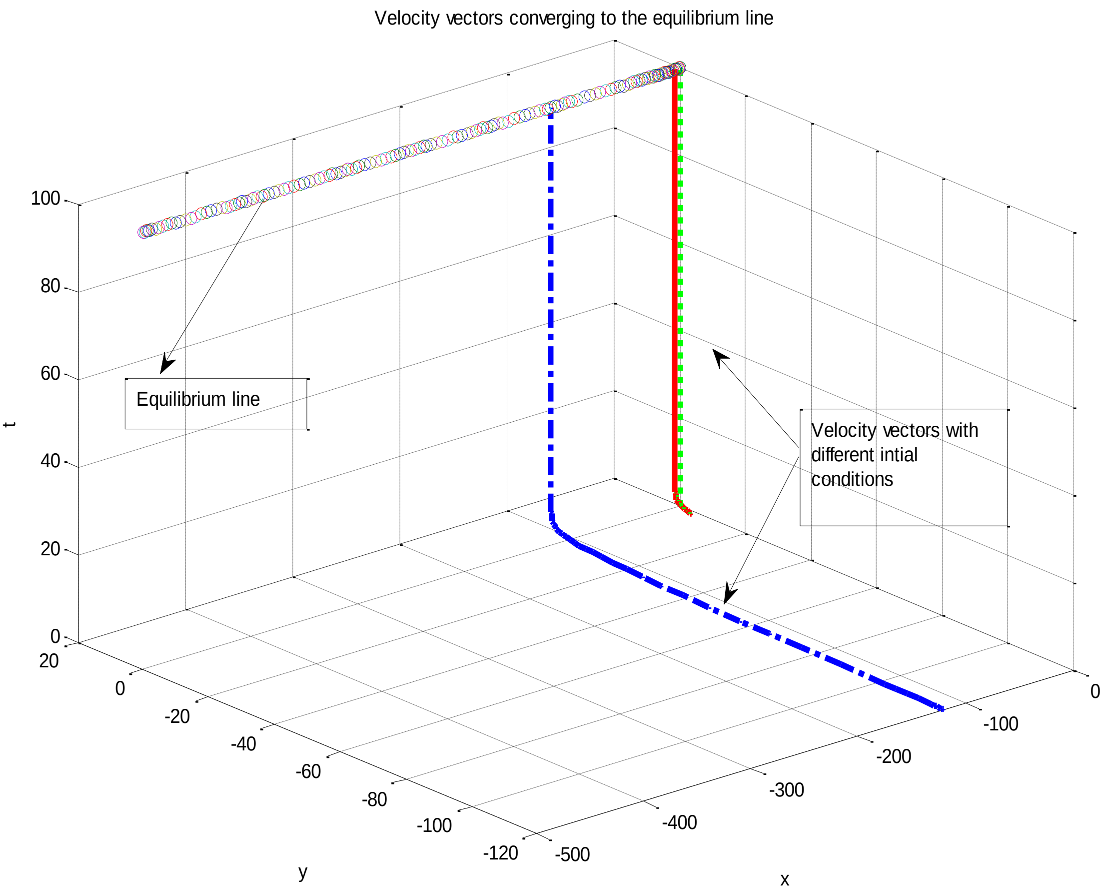
$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

In this case the general solution of the system Ax=0, where x=(x_1,x_2,x_3,x_4 )^(-1), is given by:

$$c_1\begin{pmatrix}1\\0\\0\\0\end{pmatrix}+c_2\begin{pmatrix}-1\\0\\0\\0\end{pmatrix}+c_3\begin{pmatrix}0\\0\\1\\0\end{pmatrix}+c_4\begin{pmatrix}0\\0\\-0.7071\\0.7071\end{pmatrix}e^{-t}, \quad (13)$$

wherec_1, 〖c〗_2,c_3andc_4 are arbitrary constants.

The simulation graphs of the above data for different initial positions are displayed below.




From the figure above, one can infer that, for distinct initial conditions, the velocity vectors converge to the equilibrium line given by eq. (14) below which guarantees asymptotical stability.

$$\begin{cases} x = -5\lambda \\ y = 0 \\ z = 100 \end{cases}, \quad (14)$$

Where λ∈R.

## 3. CONCLUSION

In this paper a stability analysis of dynamics of a heavy point with variable mass is made using Lyapunov direct method. In each of the different cases it is assumed that the heavy point mass is constrained to moving along the trajectory $x=\varphi(t)$ and $z=\psi(t)$vertically. Based on this trajectory different asymptotic stability conditions were constructed and numerical example verifying the applicability of the result is provided. The asymptotic stability conditions obtained in this article can be applied to different variable mass systems moving upwards in the vertical direction as it is demonstrated using examples. Moreover, as the constraining trajectories used in this article are general functions and any other convenient trajectory can be chosen as needed and stability conditions be developed in a similar manner.

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