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Solution of Ordinary Differential Equations with Variable Coefficients Using Elzaki Transform

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ABSTRACT

In this paper, we introduce a computational algorithm for solving ordinary differential equations with variable coefficients by using the modified versions of Laplace and Sumudu transforms which is called Elzaki transform. The Elzaki transform, whose fundamental properties are presented in this paper. Illustrative examples are presented to illustrate the effectiveness of its applicability.

Keywords: Elzaki transform and Differential equations with variable coefficients.

2010 Mathematics Subject Classification: 35A22, 35A24.

1. INTRODUCTION

The term "differential equations" was proposed in 1676 by G. Leibniz. The first studies of these equations were carried out in the late 17th century in the context of certain problems in mechanics and geometry. Ordinary differential equations have important applications and are a powerful tool in the study of many problems in the natural sciences and in technology; they are extensively employed in mechanics, astronomy, physics, and in many problems of chemistry and biology. The reason for this is the fact that objective laws governing certain phenomena (processes) can be written as ordinary differential equations, so that the equations themselves are a quantitative expression of these laws. For instance, Newton's laws of mechanics make it possible to reduce the description of the motion of mass points or solid bodies to solving ordinary differential equations. The computation of radio technical circuits or satellite trajectories, studies of the stability of a plane in flight, and explaining the course of chemical reactions are all carried out by studying and solving ordinary differential equations. The most interesting and most important applications of these equations are in the theory of oscillations and in automatic control theory. Applied problems in turn produce new formulations of problems in the theory of ordinary differential equations; the mathematical theory of optimal control in fact arose in this manner.

Integral transform method is widely used to solve the several differential equations with the initial values or boundary conditions, see [5-11]. In the literature there are numerous integral transforms and widely used in physics, astronomy as well as in engineering. In order to solve the differential equations, the integral transform are extensively used and thus there are several works on the theory and application of integral transform such as the Laplace, Fourier, Mellin, and Hankel, to name but a few.

Elzaki transform [1-4] which is a modified general Laplace and Sumudu transforms [1], has been shown to solve effectively, easily and accurately a large class of Linear Differential Equations. Elzaki Transform was successfully applied to integral equations, partial differential equations [2], ordinary differential equations with variable coefficients [4] and system of all these equations. The purpose of this paper is to solve Differential Equations with variable coefficients using Elzaki Transform. A new transform called the Elzaki transform defined for function of exponential order, we consider functions in the set A defined by,

$$A=\left\{f(t);\ \exists\, M, k_1, k_2 > 0, |f(t)| < Me^{\frac{|t|}{k_j}}, if\ t \in (-1)^j \times [0,\infty)\right\}. \quad (1)$$

For a given function in the set A, the constant M must be finite number, k_1, k_2 may be finite or infinite. The Elzaki transform denoted by the operator E defined by the integral equations,

$$E[f(t)] = T(v) = v\int_0^{\infty} f(t)e^{-\frac{t}{v}}dt,\ \ t \geq 0, k_1 \leq v \leq k_2, 0 \leq t < \infty\ . \quad (2)$$

For more results, theorems, various existing conditions concern to Elzaki transform, see [1-4]. Even after that, we mention here Elzaki transform of some functions, which are required in this paper.

2. ELZAKI TRANSFORM OF SOME FUNCTIONS

$f(t)$	$E[f(t)] = T(v)$
1	v^2
t	v^3
t^n	$n!\ v^{n+2}$
$\frac{t^{a-1}}{\Gamma(a)}, a > 0$	v^{a+1}
e^{at}	$\frac{v^2}{1-av}$
te^{at}	$\frac{v^3}{(1-av)^2}$
$\frac{t^{n-1}e^{at}}{(n-1)!}$, $n = 1,2,\ldots$	$\frac{v^{n+1}}{(1-av)^n}$
$sin\ at$	$\frac{av^3}{1+a^2v^2}$
$cos\ at$	$\frac{v^2}{1+a^2v^2}$
$sinh\ at$	$\frac{av^3}{1-a^2v^2}$
$cosh\ at$	$\frac{av^2}{1-a^2v^2}$
$e^{at}sin\ bt$	$\frac{bv^3}{(1-av)^2+b^2v^2}$

$e^{at}\cos bt$	$\frac{(1-av)v^2}{(1-av)^2+b^2v^2}$
$t\sin at$	$\frac{2av^4}{1+a^2v^2}$
$J_0(at)$	$\frac{v^2}{\sqrt{1+a^2v^2}}$
$H(t-a)$	$v^2e^{-\frac{a}{v}}$
$\delta(t-a)$	$ve^{-\frac{a}{v}}$

The following theorems are very useful in the study of Differential Equations with Variable Coefficients.

Theorem I

If Elzaki transform of the function $f(t)$ given by $E[f(t)] = T(v)$, then:

(i) $E[tf'(t)] = v^2\frac{d}{dv}\left[\frac{T(v)}{v} - vf(0)\right] - v\left[\frac{T(v)}{v} - vf(0)\right]$,

(ii) $E[t^2f'(t)] = v^4\frac{d^2}{dv^2}\left[\frac{T(v)}{v} - vf(0)\right]$,

(iii) $E[tf''(t)] = v^2\frac{d}{dv}\left[\frac{T(v)}{v^2} - f(0) - vf'(0)\right] - v\left[\frac{T(v)}{v^2} - f(0) - vf'(0)\right]$,

(iv) $E[t^2f''(t)] = v^4\frac{d^2}{dv^2}\left[\frac{T(v)}{v^2} - f(0) - vf'(0)\right]$.

Proof

To prove (i) we proceed the following way:

Let $T(v)$ be Elzaki transform of the function $f(t)$ in A, then the function $tf(t)$ is in A since $f(t)$ is so.

We now find $E(tf(t))$ and integrating by parts we find that

$$\frac{d}{dv}T(v) = T'(v) = \frac{d}{dv}\int_0^\infty ve^{-\frac{t}{v}}f(t)dt = \int_0^\infty \frac{\partial}{\partial v}\left[ve^{-\frac{t}{v}}f(t)\right]dt$$

$$= \int_0^\infty e^{-\frac{t}{v}}(tf(t))dt + \int_0^\infty e^{-\frac{t}{v}}f(t)dt = \frac{1}{v^2}E[tf(t)] + \frac{1}{v}E[f(t)]$$

$$\therefore \frac{d}{dv}T(v) = \frac{1}{v^2}E[tf(t)] + \frac{1}{v}E[f(t)].$$

Hence we obtain $E[tf(t)] = v^2\frac{d}{dv}T(v) - vT(v)$ (3)

Now putting $f(t) = f'(t)$ in (3) we have,

$$E[tf'(t)] = v^2 \frac{d}{dv}\left[E\left(f'(t)\right)\right] - vE[f'(t)]$$

$$= v^2 \frac{d}{dv}\left[\frac{T(v)}{v} - vf(0)\right] - v\left[\frac{T(v)}{v} - vf(0)\right]$$

$$\Rightarrow E[f'(t)] = \left[\frac{T(v)}{v} - vf(0)\right].$$

To prove $(\boldsymbol{ii})$ we use eq. (3) ,

Now putting $f(t) = tf'(t)$ in (3) we have

$$E[t^2 f'(t)] = v^2 \frac{d}{dv}\left[E\left(tf'(t)\right)\right] - vE[tf'(t)]$$

$$= v^2 \frac{d}{dv}\left\{v^2 \frac{d}{dv}\left[\frac{T(v)}{v} - vf(0)\right] - v\left[\frac{T(v)}{v} - vf(0)\right]\right\} - v\left\{v^2 \frac{d}{dv}\left[\frac{T(v)}{v} - vf(0)\right] - v\left[\frac{T(v)}{v} - vf(0)\right]\right\}$$

$$= 2v^3 \frac{d}{dv}\left[\frac{T(v)}{v} - vf(0)\right] + v^4 \frac{d^2}{dv^2}\left[\frac{T(v)}{v} - vf(0)\right] - v^2\left[\frac{T(v)}{v} - vf(0)\right] - 2v^3 \frac{d}{dv}\left[\frac{T(v)}{v} - vf(0)\right] + v^2\left[\frac{T(v)}{v} - vf(0)\right]$$

$$= v^4 \frac{d^2}{dv^2}\left[\frac{T(v)}{v} - vf(0)\right].$$

Similarly we can prove (iii) and (iv) in the same manner of the above proving procedures and steps of (i) and (ii).

3. APPLICATIONS

In this section, the effectiveness and the usefulness of Elzaki transform are demonstrated by finding exact solutions of ordinary differential equations with variable coefficients with graph.

Example 3.1.

Solve the differential equation:

$$ty'' + y' + 4ty = 0, \qquad y(0) = 3,\ \ y'(0) = 0,\ \ t > 0.$$

By Using Elzaki transformation into the given equation and Theorem I, we have

$$v^2 \frac{d}{dv}\left[\frac{E(y)}{v^2} - y(0) - vy'(0)\right] - v\left[\frac{E(y)}{v^2} - y(0) - vy'(0)\right] + \frac{E(y)}{v} - vy(0) + 4\left[v^2 \frac{d}{dv}E(y) - vE(y)\right] = 0$$

Using the Initial conditions we get,

$$E'(y) - \frac{(2+4v^2)}{v(1+4v^2)} E(y) = 0.$$

This is a Linear Differential Equation for unknown function E, having the solution in the form

$$E(y) = \frac{cv^2}{\sqrt{1+4v^2}}$$

by Using the Elzaki Inverse transform we obtain the Solution

$$y = cE^{-1}\frac{v^2}{\sqrt{1+4v^2}}$$

$$y(t) = cJ_o(2t). \qquad (4)$$

We now determine ‘c’ by applying the initial condition i.e.,

$$y(0) = cJ_o(0)$$

$$\Rightarrow c = 3$$

Putting the value of ‘c’ in (4) we obtain the solution in the form of

$$y = 3J_o(2t). \qquad (5)$$

Fig. 1.1 gives the graphical representation of (5)

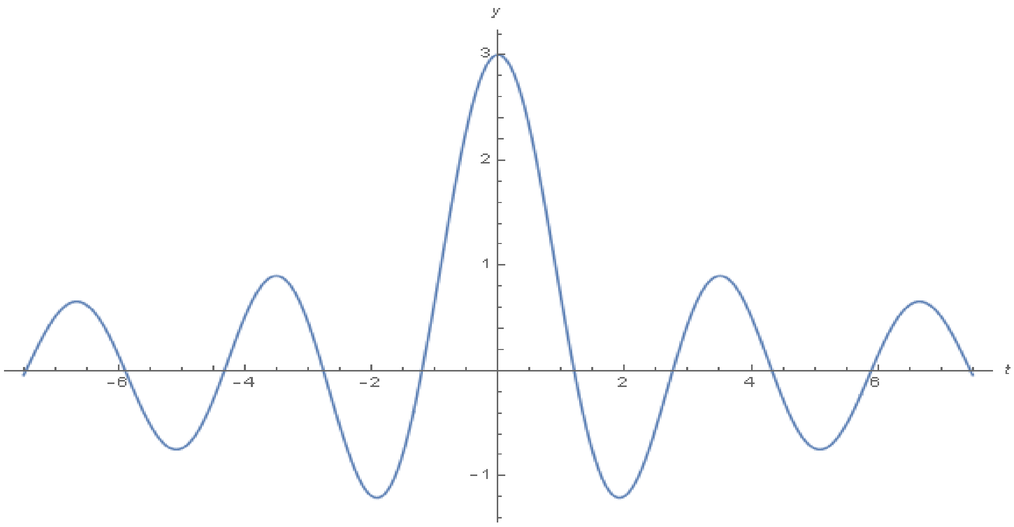

Fig. 1.1

Example 3.2

Solve the differential equation:

$$ty'' - ty' + y = 2, \qquad y(0) = 2, \ \ y'(0) = -4, t > 0.$$

By using Elzaki Transformation into the given equation and Theorem I, we have

$$v^2\frac{d}{dv}\left[\frac{E(y)}{v^2}-y(0)-vy'(0)\right]-v\left[\frac{E(y)}{v^2}-y(0)-vy'(0)\right]-v^2\frac{d}{dv}\left[\frac{E(y)}{v}-vy(0)\right]+v\left[\frac{E(y)}{v}-vy(0)\right]$$
$$+E(y)=2v^2$$

Using the Initial conditions we get

$$E'(y)-\frac{3}{v}E(y)+2v=0.$$

This is a Linear Differential Equation for unknown function E, having the solution in the form

$$E(y)=2v^2+cv^3$$

By using the Elzaki Inverse Transform we obtain the Solution in the following way:

$$y(t)=2E^{-1}(v^2)+cE^{-1}(v^3)$$
$$y(t)=2+ct. \qquad (6)$$

We now determine 'c' by applying the Initial Condition

$$y'(t)=c$$
$$\therefore y'(0)=c$$
$$-4=c$$

Putting the value of 'c' in (6) we obtain the solution in the form of

$$y=2-4t. \qquad (7)$$

Example 3.3

Solve the differential equation:

$$t^2y'+2ty=\sinh t, \quad y(0)=\frac{1}{2}, \quad t>0.$$

By using Elzaki transformation into the given equation and Theorem I, we have

$$v^4\frac{d^2}{dv^2}\left[\frac{E(y)}{v}-vy(0)\right]+2\left[v^2\frac{d}{dv}E(y)-vE(y)\right]=\frac{v^3}{1-v^2}$$

Using the Initial Conditions we get

$$E''(y)=\frac{1}{1-v^2}$$

The Solution of this Equation is

$$E(y) = C_1 + C_2 v + \sum_{i=0}^{\infty} \frac{v^{2i+2}}{(2i+1)(2i+2)}$$

Substituting the condition into the above equation we get:

$$E(y) = \sum_{i=0}^{\infty} \frac{v^{2i+2}}{(2i+1)(2i+2)}$$

By using Elzaki Inverse Transform we obtain the Solution

$$y(t) = E^{-1}\left(\sum_{i=0}^{\infty} \frac{v^{2i+2}}{(2i+1)(2i+2)}\right)$$

$$y(t) = \sum_{i=0}^{\infty} \frac{t^{2i}}{(2i+2)!} = \frac{1}{2} + \frac{t^2}{4!} + \frac{t^4}{6!} + \cdots$$

$$y(t) = \frac{\cosh(t) - 1}{t^2}. \qquad (8)$$

Fig. 1.2 gives the graphical representation of (8)

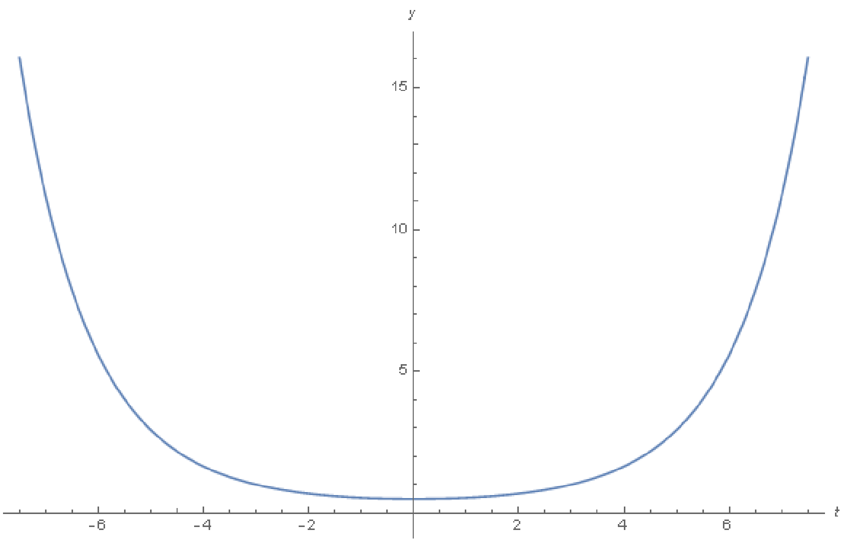

Fig. 1.2

Example 3.4

Solve the differential equation:

$$ty'' - 2ty' + y = 0, \qquad y(0) = 0, \;\; y'(0) = -1, t > 0$$

By using Elzaki Transformation into the given equation and Theorem I, we have

$$v^2\frac{d}{dv}\left[\frac{E(y)}{v^2}-y(0)-vy'(0)\right]-v\left[\frac{E(y)}{v^2}-y(0)-vy'(0)\right]-2v^2\frac{d}{dv}\left[\frac{E(y)}{v}-vy(0)\right]+2v\left[\frac{E(y)}{v}-vy(0)\right]$$
$$+4E(y)=0$$

Using the Initial Conditions we get

$$E'(y)+\frac{8v-3}{v(1-2v)}E(y)=0$$

This is a Linear Differential Equation for unknown function E, having the solution in the form

$$E(y)=c(v^3-2v^4)$$

By using the Elzaki Inverse Transform we obtain the Solution in the following way:

$$y(t)=cE^{-1}(v^3)-2cE^{-1}(v^4)$$
$$y(t)=c(t-t^2) \qquad (9)$$

We now determine 'c' by applying the Initial Condition i.e.,

$$y'(t)=c(1-2t)$$
$$\therefore y'(0)=c(1-0)$$
$$-1=c$$

Putting the value of 'c' in (6) we obtain the solution in the form of

$$y=t^2-t. \qquad (10)$$

4. CONCLUSION

In this work, Elzaki transform is applied to obtain the solution of ordinary differential equations with variable coefficients. It may be concluded that Elzaki transform is very powerful and efficient in finding the solution for a wide class of ordinary differential equations with variable coefficients.

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