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Observation on the Elliptic Paraboloid $x^2 + y^2 = 19z$

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ABSTRACT

This paper concerns with the problem of obtaining a general solution of the ternary quadratic equation $x^2 + y^2 = 19z$ based on its given initial solution.

Keywords: Ternary quadratic, Elliptic paraboloid and Integer solutions.

1. INTRODUCTION

The quadratic Diophantine equations with three unknowns offer an unlimited field for research because of their variety [1-3]. For an extensive review of various problems on ternary quadratic Diophantine equations representing specific 3 dimensional surfaces, one may refer [4-24]. In this communication, we present a problem of obtaining a general solution of the equation $x^2 + y^2 = 19z$ based on its given initial solution.

2. METHOD OF ANALYSIS

The ternary quadratic Diophantine equation under consideration is,

$$x^2 + y^2 = 19z \qquad (1)$$

To start with, it is seen that equation (1) is satisfied by the triples of integers $(19\alpha, 19\alpha, 38\alpha^2)$,

$(38mn, 19(m^2 - n^2), 19(m^2 + n^2)^2)$,

$(19m(m^2 + n^2), 19n(m^2 + n^2), 19(m^2 + n^2)^3)$,

$(19(m^3 - 3mn^2), 19(3m^2 n - n^3), 19(m^2 + n^2)^3)$

A natural question that arises now is that, whether a general formula for obtaining a sequence of integer solutions for (1) based on its given integer solution can be found? The answer to the question is yes and a method of obtaining the same is illustrated below:

Let (x_0, y_0, z_0) be the given integer solution to (1). Let (x_1, y_1, z_1) be the first solution of (1) where,

$$x_1 = 4h_0 - x_0,\ y_1 = 2h_0 - y_0,\ z_1 = z_0 + h_0^2 \qquad (2)$$

Where h is any non-zero integer to be determined.
Substituting (2) in (1) and simplifying, we have

$$h_0 = 8x_0 + 4y_0 \qquad (3)$$

Using (3) in (2) we have,

$$x_1 = 31x_0 + 16y_0,\ y_1 = 16x_0 + 7y_0,\ z_1 = z_0 + (8x_0 + 4y_0)^2$$

Repeating the above process, the general solution (x_n, y_n, z_n) of (1) is represented by,

$$x_n = \frac{1}{5}\left[\left(4(39)^n + (-1)^n\right)x_0 + 2\left(39^n - (-1)^n\right)y_0\right]$$

$$y_n = \frac{1}{5}\left[2\left(39^n - (-1)^n\right)x_0 + \left(39^n + 4(-1)^n\right)y_0\right]$$

$$z_n = z_0 + \frac{1}{95}\left(39^{2n} - 1\right)(2x_0 + y_0)^2$$

A few interesting relations among the solutions are exhibited as follows:

- $\frac{6(y_{2n} + 2x_{2n})}{y_0 + 2x_0}$ is a Nasty number
- $\frac{6(x_{2n} - 2y_{2n})}{x_0 - 2y_0}$ is a Nasty number
- $6\{(y_n + 2x_n)^2 - 95(z_n - z_0)\}$ is a Nasty number
- $x_n + x_{n+2} + 2x_{n+1} = 640(y_n + 2x_n)$
- $y_n + y_{n+2} + 2y_{n+1} = 320(y_n + 2x_n)$
- $x_n + x_{n+2} + 2x_{n+1} = 2(y_n + y_{n+2} + 2y_{n+1})$
- $x_{n+1} = 16y_n + 31x_n$
- $y_{n+1} = 7y_n + 16x_n$
- $y_{n+1} + y_{n+2} = 312(y_n + 2x_n)$
- $x_{n+1} + x_{n+2} = 624(y_n + 2x_n)$

Remark:

It is worth to mention here that the general solution obtained above for equation (1) is not unique. In particular, we have two more choices of general solutions to equation (1) that are presented below. Let (x_0, y_0, z_0) be any given solution of (1)

Choice: 1

Consider the transformation,

$$x_1 = 3h_0 + x_0,\ y_1 = 3h_0 + y_0,\ z_1 = z_0 + h_0^2$$

Following the above process, the general solution is,

$$x_n = \frac{1}{2}\left[\left(37^n + 1\right)x_0 + \left(37^n - 1\right)y_0\right]$$

$$y_n = \frac{1}{2}\left[\left(37^n - 1\right)x_0 + \left(37^n + 1\right)y_0\right]$$

$$z_n = z_0 + \left(x_0 + y_0\right)^2 \frac{\left(37^{2n} - 1\right)}{38}$$

A few interesting relations among the solutions are exhibited as follows:

- $6\left\{\left(x_n + y_n\right)^2 - 38\left(z_n - z_0\right)\right\}$ is a Nasty number
- $6\left\{38\left(z_n - z_0\right) + \left(x_0 + y_0\right)^2\right\}$ is a Nasty number
- $x_n^2 - y_n^2 = 37^n\left(x_0^2 - y_0^2\right)$
- $2\left(x_n y_n - x_0 y_0\right) = 19\left(z_n - z_0\right)$
- $x_{n+1} = 19x_n + 18y_n$

Choice: 2

Consider the transformation,

$$x_1 = 4h_0 + x_0,\ y_1 = h_0 + y_0,\ z_1 = z_0 + h_0^2$$

The general solution is,

$$x_n = \frac{1}{17}\left[\left(16\left(18^n\right) + 1\right)x_0 + 4\left(18^n - 1\right)y_0\right]$$

$$y_n = \frac{1}{17}\left[4\left(18^n - 1\right)x_0 + \left(18^n + 16\right)y_0\right]$$

$$z_n = z_0 + \left(4x_0 + y_0\right)^2 \frac{\left(18^{2n} - 1\right)}{323}$$

A few interesting relations among the solutions are exhibited as follows:

- $6\left\{\left(4x_n + y_n\right)^2 - 323\left(z_n - z_0\right)\right\}$ is a Nasty number
- $6\left\{323\left(z_n - z_0\right) + \left(4x_0 + y_0\right)^2\right\}$ is a Nasty number
- $x_{n+1} = 17x_n + 4y_n$
- $y_{n+1} = 4x_n + 2y_n$
- $x_{n+1} + x_{n+2} = 322x_n + 80y_n$

3. CONCLUSION

In this paper, general formulas for generating a sequence of solutions based on the given solution of the elliptic paraboloid $x^2 + y^2 = 19z$ are obtained. One may attempt to find general formulas for other choices of the elliptic paraboloid.

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